A study of multivariate pre-control charts

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Abstract

The quality of the output of a production process is often measured by the joint level of several correlated characteristics. Through multivariate control charts, one will be able to detect a process change and prevent defects from occurring by identifying and eliminating assignable causes of variation. In contrast to the traditional control charts, pre-control charts focus on evaluating the process capability during the set-up stage and detecting the process change during the mass production stage. However, the set-up and monitoring rules as well as the sample size for multivariate pre-control charts have not been thoroughly studied (to our knowledge). The main purpose of this research is to develop these rules and compare the performances of detecting a process change using multivariate pre-control charts versus Hotelling $T^2$ control charts when the quality characteristics follow a multivariate normal distribution. These objectives can be achieved by two statistical measures known as the in-control and out-of-control average run lengths (ARLs). The simulation results and a numerical example further demonstrate the usefulness of the new set-up and monitoring rules we proposed for multivariate pre-control charts.

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Keywords: Multivariate Pre-control charts; Hotelling $T^2$ control charts; Average run lengths

1. Introduction

Statistical quality control charts are widely used in industries. In contrast to the traditional $\bar{X} - R$ control charts, pre-control charts focus on evaluating the process capability during the set-up stage and detecting the process change during the mass production stage. Lower operating cost and ease of use (savings in time) are two advantages of using pre-control charts as a process assessment and monitoring tool. Moreover, the quality of the output of a modern day production process is often measured by the joint level of several correlated characteristics. For example, the tensile strength and diameter of a glass fiber are two important quality characteristics that are to be jointly controlled. The importance of statistical process control (SPC) for multivariate quality characteristics as well as multivariate SPC charts has been increasingly recognized by academics and practitioners alike, as witnessed by recent issues in journals such as International Journal of Productions Economics (Villalobos et al., 2005; Yang and Rahim, 2005), Journal of Quality Technology (Kim and Reynolds, 2005). Through multivariate control charts, one will be able to detect the process change and prevent defects from occurring by identifying and eliminating assignable causes of variation. However, the set-up and monitoring rules for multivariate...
pre-control charts have not been thoroughly studied (to our knowledge). Thus, the main purpose of this research is to develop these rules and compare the performances of detecting process change using multivariate pre-control charts versus Hotelling $T^2$ control charts when quality characteristics follow multivariate normal distributions. The objectives of this research are listed as follow:

1. Evaluate the current set-up and monitoring rules for pre-control charts under a normal distribution.
2. Develop the new set-up and monitoring rules for multivariate pre-control charts and compare them with the current one.
3. Compare the performances of the multivariate pre-control chart versus the Hotelling $T^2$ control chart.

2. Literature review

Satterthwaite (1954) established the first simple and effective univariate pre-control chart as shown in Fig. 1, where USL is the upper specification limit, LSL is the lower specification limit, UPCL is the upper pre-control limit, LPCL is the lower pre-control limit, $T$ is the Target, the green zone represents the probability of $Pr(LPCL < x < UPCL)$, the yellow zone represents the probability of $Pr(LPCL < x < USL) + Pr(LSL < x < LPCL)$ and the red zone represents the probability of $Pr(USL < x) + Pr(LSL > x)$.

According to Bohle (1988) and Ledolter and Swersey (1997), the set-up and monitoring rules of the above univariate pre-control chart can be stated as follows:

1. Assume that the output of the process follows a normal distribution.
2. The center of the specification is the target.
3. The pre-control lines are set in the middle of the lower/upper specification limits (USL/LSL) and the centerline. The area between the two pre-control lines is called the green zone; the area between USL/LSL and the two pre-control lines is called the yellow zone; the area that exceeds the USL or LSL is called the red zone.
4. The set-up rule for pre-control charts is to assess the process capability prior to a mass production. Once a machine/process is set-up, the operator will take five consecutive samples. If all the five samples fall in the green zone, the process set-up is considered to be passed and a mass production can be run afterwards. Otherwise, the set-up needs to be retuned/readjusted.
5. During the mass production, the following monitoring rules are used by the operators:

(1) Consecutively take two samples to inspect. If both samples fall in the green zone, which indicates that the process is stable; then the mass production can be continued.
(2) If one sample falls in the Green zone and the other falls in the yellow zone; the mass production can still be continued.
(3) If both samples fall in the yellow zone, which indicates that the process has shifted (both fall in the same zone) or a large process variation exists (one falls in the upper zone, while the other in the lower zone); then the process needs to be stopped for readjustment.
(4) If any sample falls in the red zone, the process must be stopped to find out the causes of the problem.

The above set-up and monitoring rules have also been referred by Ermer and Roepke (1991), Logothetis (1990), Mackertich (1990), and Traver (1985).

Hotelling (1947), Tracy et al. (1992), Prins and Mader (1997), Khoo and Quah (2002) proposed various multivariate control charts. Once the data collection of several correlated characteristics in a multivariate control chart is completed, several multivariate process capability indices such as Taam et al. (1993), Chen (1994), Pan and Lee (2003) can be calculated accordingly. However, the above
process monitoring and capability assessment require a sophisticated automatic system for data collection and an extensive training time and effort. To facilitate the use of a multivariate control chart at the operator level, Hubele (1986) developed the multivariate pre-control chart for a multivariate normal distribution as follows:

Let $\mathbf{X}$ be a vector consisting of $p$ components and the distribution function $F(x)$ is given. Suppose that $x_1, x_2, \ldots, x_p$ are $p$ random variables with mean value equals $\mu$ and covariance matrix equals $\Sigma$, then the mathematical symbols for deriving a multivariate pre-control control chart can be denoted as follow:

$S_i$: The specification function of a given design;

$DS_i$: The defining domain of $S_i$ for each $x_i$;

$N_i$: The narrow limiting gauge function;

$DN_i$: The defining domain of $N_i$ for each $x_i$.

$$P_s = \int_{DS_1} \int_{DS_2} \cdots \int_{DS_p} dF(x) \leq 1,$$

$$P_N = \int_{DN_1} \int_{DN_2} \cdots \int_{DN_p} dF(x) \leq 1,$$

$$DN_i \subseteq DS_i \Rightarrow P_N < P_S.$$

Suppose that $F(x)$ is a multivariate normal distribution and according to the test statistics of a multivariate pre-control chart proposed by Hotelling (1947):

$$T^2 = (\mathbf{X} - \mu)^\Sigma^{-1}(\mathbf{X} - \mu) \leq \chi^2_p(z),$$

where (1) follows a Chi-square distribution with $df = p$. It is an ellipse with probability value $1-\alpha$.

Using the above properties of the multivariate statistics and the pre-control concepts in which the probabilities of specifications and narrow limiting zones are $P_N = 1 - z_N$ and $P_S = 1 - z_S$ respectively, Hubele (1986) (Fig. 2) designed the multivariate pre-control chart, of which the three different color zones as shown in Fig. 2 are defined as:

Green zone: $\{X|(X - \mu)^\Sigma^{-1}(X - \mu) \leq \chi^2_p(z_N)\}$,

Yellow zone: $\{X|(X - \mu)^\Sigma^{-1}(X - \mu) > \chi^2_p(z_N) \cap (X - \mu)^\Sigma^{-1}(X - \mu) \leq \chi^2_p(z_S)\}$,

Red zone: $\{X|(X - \mu)^\Sigma^{-1}(X - \mu) > \chi^2_p(z_S)\}$.

3. A multivariate pre-control chart for a normal distribution

Since the set-up and monitoring rules of the multivariate pre-control chart proposed by Hubele (1986) are similar to those of the univariate pre-control charts, the suitability of these rules is very questionable. In this paper, the new set-up and monitoring rules for the multivariate pre-control charts for a normal distribution will be developed. We also compare the performances of detecting the process change using the Hotelling $T^2$ versus the multivariate pre-control charts.

3.1. Determination of the sample size of the set-up rule for a multivariate pre-control chart

According to Ledolter and Swersey (1997), the criterion of set-up rule for multivariate pre-control charts can be derived as follow:

Assume that a process fallout follows a normal distribution and let $C_p = c$. If the process center is 0 with USL/LSL = $\pm 2$, then the pre-control limits are $\pm 1$. Based on this assumption, one can calculate the probability of passing the set-up rule as follows:

Let $C_p = \frac{USL - LSL}{6\sigma} = c$,

then $\frac{1}{6\sigma} = c$ and $\sigma = \frac{1}{6c}$.

The probability of passing the set-up rule $= \{Pr[|Z| \leq 1]\}^5 = \{Pr[|Z| \leq (1.5)c]\}^5$.

Based on (2), one can obtain the probability of qualifying the process during the set-up stage under
different process capability indices as shown in Table 1.

Table 1 shows that a larger \( C_p \) leads to a higher probability of passing the set-up rule. However, the probability of accepting/rejecting the set-up rule equals 0.488 when \( C_p = 1 \). Thus, it is very likely to have a false judgment when \( C_p = 1 \). In order to reduce the risk of a false judgment, one can increase the sample size from 5 to 12, then the probability of a Type-I error or the risk of a false-judgment of passing the set-up rule will be reduced from 0.488 to 0.178 (when \( C_p = 1 \)).

We first calculate the probabilities of passing the set-up rule under different combinations of sample sizes and process capability indices as shown in Table 2.

Generally speaking, the risk of a false judgment can be reduced by adding more samples. As shown in Table 2, the probability of a Type I error can be reduced from 0.488 to 0.178 if the sample size increases from 5 to 12 when \( C_p = 1 \). However, the marginal effect will be gradually reduced as the sample size continues to increase. Table 3 shows the marginal decrease of the probability of passing the set-up rule by adding one sample each time under different process capability indices.

Table 3 also indicates that if we increase the sample size from 10 to 11 when \( C_p = 1 \), then the marginal decrease of the probability of passing the set-up rule will be 0.0318. From Table 2, on the other hand, the probability of passing the set-up rule is equal to 0.238 when sampling 10 units at \( C_p = 1 \). In other words, the confidence level will be closed to 0.8 with a Type I error = 0.238 if the sample size equals 11 and 12 respectively, which means Type I error will be closed to 0.2 when the sample size is greater than 10. Thus, we propose to use a sample size of 10 for the new set-up rule.

The new multivariate process capability index \( \text{NMC}_p \) proposed by Pan and Lee (2003) is

\[
\text{NMC}_p = \text{vol}(R_1^*)/\text{vol}(R_2), \quad \text{where} \quad R_1^* = |A^*|^{1/2}(9\pi)^{v/2}[\Gamma(v/2 + 1)]^{-1} \quad \text{is a revised specification zone (an elliptical area adjusted by the correlation coefficient)}, \quad A^* \quad \text{is the adjusted correlation matrix for a stable process,} \quad v \quad \text{is the number of correlated characteristics and} \quad R_2 = |S|^{1/2}(\pi K)^{v/2}[\Gamma(v/2 + 1)]^{-1} \quad \text{is the process zone with 99.73% confidence level,} \quad S \quad \text{is the sample correlation matrix,} \quad K \quad \text{is the chi-square value with the degree of freedom equals} \quad v. \quad \text{When the sample size is} \quad 5 \quad \text{and the multivariate process capability index equals} \quad 1, \quad \text{Pan and Lee (2003) pointed out that} \quad \text{the probability of passing the set-up rule} = (0.86)^5 = 0.470. \quad \text{The risk of a false-judgment or} \quad \text{the probability of passing the set-up rule will be reduced to} \quad (0.86)^{10} = 0.22 \quad \text{if the sample size is increased to} \quad 10. \quad \text{In other words, the confidence level will be closed to} \quad 0.8 \quad \text{if the sample size is increased to}
\]
10 during the set-up stage of using multivariate pre-control charts. It further suggests to use a sample size of 10 for the new set-up rule at a 80% confidence level.

3.2. Different monitoring rules for multivariate pre-control charts

3.2.1. Introduction of new monitoring rules and pre-control conditions

Hubele (1986) sets the green zone of a bivariate pre-control chart to be 0.86 of the process zone, yellow zone to be between 0.86 and 0.999 of the process zone, and red zone to be beyond the 0.999 of the process zone. In reality, the green, yellow and red zones can be determined from the Type I error (false alarm rate) specified by the producer. When the green zone is 0.86 of the process zone, its Type I error \( \alpha \) equals approximately 2%. If the producer wants to reduce the false alarm rate to 1%, then the green zone needs to be adjusted from 0.86 to 0.9. However, if a 3% false alarm rate can be tolerated by the producer, then the green zone needs to be adjusted from 0.86 to 0.83. Hence, the three pre-control conditions are:

1. Keep green zone at 0.86 (\( \alpha = 0.02 \)).
2. Adjust green zone from 0.86 to 0.9 (\( \alpha = 0.01 \)).
3. Adjust green zone from 0.86 to 0.83 (\( \alpha = 0.03 \)).

To compare the performances of various multivariate pre-control charts with different monitoring rules under the above three pre-control conditions, we increase the sample size of the monitoring rule from 2 to 3, 4 and 5 (i.e. about half of the sample size required during the set-up stage) for reducing the risk of false judgment when \( C_p = 1 \), then the following six new monitoring rules are:

Rule #1(sample size = 3): 3 green, 2 green 1 yellow, 1green 2 yellow, indicates the process is stable.
Rule #2(sample size = 3): 3 green, 2 green 1 yellow, indicates the process is stable; if 1 green 2 yellow occurs, then one more sample needs to be added.
Rule #3(sample size = 3): 3 green, 2 green 1 yellow, indicates the process is stable; if 1 green 2 yellow occurs, then two more samples need to be added.
Rule #4(sample size = 4): 4 green, 3 green 1 yellow, 2 green 2 yellow, indicates the process is stable.
Rule #5(sample size = 4): 4 green, 3green 1 yellow, indicates the process is stable; if 2 green 2 yellow occurs, then one more sample needs to be added.

Rule #6(sample size = 5): 5 green, 4 green 1 yellow, 3 green 2 yellow, indicates the process is stable.

The above six new monitoring rules will be compared with the original one under the various pre-control conditions in order to determine the most appropriate one.

3.2.2. Calculation of Type I and II errors for the pre-control chart

If we use the original monitoring rule and the pre-control condition proposed by Hubele (1986), the probability of a type I error for the monitoring rule can be calculated as follows: Assume that \( H_0 \) indicates the process is stable and \( H_1 \) indicates the process is unstable, then the false alarm rate of the pre-control chart equals:

\[
x = P(\text{reject } H_0 | H_0) = 1 - 0.7396 - 0.23908 = 0.02132,\]

where the probability of two samples falling in the the green zone equals 0.86 \( \times \) 0.86 = 0.7396, and the probability of one green, one yellow equals \( 2 \times 0.86 \times (0.999 - 0.86) = 0.23908 \).

Due to the difficulties of calculating the Type II error \( \beta \), we use a computer to simulate 10,000 data with \((b_1, b_2)\) process mean shifts as indicated in Section 3.2.4 and apply the similar acceptance criteria as indicated in Section 2 for the original monitoring rule, then the probability of a Type II error can be computed as

\[
\beta = P(\text{accept } H_0 | H_1) = \text{numbers of false judgment}/10,000.
\]

3.2.3. Criteria of comparison for different monitoring rules

In order to compare the performances of the set-up and monitoring rules for different monitoring rules of multivariate pre-control charts, we use the average run lengths (ARLs) as the criteria for assessing the performances of different monitoring rules. The in-control ARL is defined as \( 1/\alpha \). Thus, it is desirable that the average run length between two consecutive false alarms is longer if the process is in-control. The out-of-control ARL is defined as \( 1/(1-\beta) \). Thus, it is desirable that the average run length between two consecutive false alarms is shorter if the process is out-of-control.
### 3.2.4. A comparison of different monitoring rules under various pre-control conditions

Without loss of generality, we assume that the process data follows a bivariate normal distribution with $\mu = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ and $\Sigma = \left( \begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array} \right)$. The in-control and out-of-control ARLs of the various pre-control charts under low correlation ($\rho = 0.2$), moderate correlation ($\rho = 0.5$) and high correlation ($\rho = 0.8$) are thoroughly discussed in this section. Three different pre-control conditions as stated in Section 3.2.1 as well as four different process mean shifts of $(b_1, b_2)$ such as two processes without mean shifts, two small mean shifts, one large and one small mean shift, and two large mean shifts as indicated from Tables 4 to 8 will also be considered. In order to obtain the average in-control and out-of-control ARLs under various conditions, a simulation study using R program (a statistical computing software developed by GNU/Linux system) is conducted for at least 10,000 times.

If one sets the green zone of a bivariate pre-control chart to be 0.86 of the process zone, the yellow zone to be between 0.86 and 0.999 of the process zone, and the red zone to be beyond 0.999 of the process zone, then the simulation results of ARLs under different monitoring rules for the various process mean shifts when $\rho = 0.2$ are shown in Table 4.

When two quality characteristics are lowly correlated, Table 4 shows that Rules #1–4 performs better in terms of the in-control ARLs. Moreover, Rule #3 performs the best in terms of the out-of-control ARLs among these four rules. The performance of rule #3 is worse than the original one only when both processes have small shifts, but its in-control ARL is better than the original one. Rules #5 and #6 have better out-of-control ARLs, however, their in-control ARLs are worse than Rule #3 and the original one. If one sets the green zone of a bivariate pre-control chart to be 0.9 of the process zone, the yellow zone to be between 0.9 and 0.999 of the process zone, and the red zone to be beyond 0.999 of the process zone, then the simulation results of ARLs under different monitoring rules for the various process mean shifts when $\rho = 0.2$ are shown in Table 5.

When two quality characteristics are lowly correlated, Table 5 shows that Rules #1, 2, 3 and 4 perform better in terms of the in-control ARLs. Moreover, Rule #3 performs the best in terms of the out-of-control ARLs among these four rules. The performance of Rule #3 is worse than the original one only when both processes have small shifts, but its in-control ARL is better than the original one.

<table>
<thead>
<tr>
<th>$(b_1, b_2)$</th>
<th>Original</th>
<th>Rule #1</th>
<th>Rule #2</th>
<th>Rule #3</th>
<th>Rule #4</th>
<th>Rule #5</th>
<th>Rule #6</th>
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<td>32.72</td>
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**Notes:**

1. $(b_1, b_2)$ indicates the amount of process shift in terms of standard deviations, e.g. (0.5, 0.5) Indicates that both the means of the two processes have shifted by 0.5 standard deviation.
2. The boldfaced numbers indicate that their ARLs are better than those of the original ones.
Rules #5 and #6 have better out-of-control ARLs, however, their in-control ARLs are worse than Rule #3 and the original one. If one sets the green zone of a bivariate pre-control chart to be 0.83 of the process zone, the yellow zone to be between 0.83 and 0.999 of the process zone, and the red zone to be beyond 0.999 of the process zone, then the simulation results of ARLs under the different
monitoring rules for various process mean shifts when $r = 0.2$ are shown in Table 6.

When two quality characteristics are lowly correlated, Table 6 shows that Rules #1, 2, 3 and 4 perform better in terms of the in-control ARLs. Moreover, Rule #3 performs the best in terms of the out-of-control ARLs among these four rules.

### Table 7
A comparison of ARLs under different monitoring rule for the various process mean shifts when $r = 0.5$ (pre-control condition #3)

<table>
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<td>1.22</td>
<td>1.29</td>
<td>1.11</td>
<td>1.10</td>
</tr>
<tr>
<td>(2,2)</td>
<td>2.90</td>
<td>4.03</td>
<td>2.23</td>
<td>1.86</td>
<td>2.16</td>
<td>1.57</td>
<td>1.54</td>
</tr>
</tbody>
</table>

**Notes:**
1. $(b_1, b_2)$ indicates the amount of process shift in terms of standard deviations, e.g. (0.5, 0.5) indicates that both the means of the two processes have shifted by 0.5 standard deviation.
2. The boldfaced numbers indicate that their ARLs are better than those of the original ones.

### Table 8
A comparison of ARLs under different monitoring rules for the various process mean shifts when $r = 0.8$ (pre-control condition #3)

<table>
<thead>
<tr>
<th>$(b_1, b_2)$</th>
<th>Original</th>
<th>Rule #1</th>
<th>Rule #2</th>
<th>Rule #3</th>
<th>Rule #4</th>
<th>Rule #5</th>
<th>Rule #6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>32.72</td>
<td>127.82</td>
<td>50.22</td>
<td>33.39</td>
<td>48.00</td>
<td>24.45</td>
<td>23.94</td>
</tr>
<tr>
<td>(0.25,0.25)</td>
<td>32.34</td>
<td>125.63</td>
<td>50.51</td>
<td>32.30</td>
<td>45.37</td>
<td>23.91</td>
<td>23.11</td>
</tr>
<tr>
<td>(0.25,−0.25)</td>
<td>28.57</td>
<td>97.09</td>
<td>40.36</td>
<td>26.95</td>
<td>39.18</td>
<td>20.67</td>
<td>19.94</td>
</tr>
<tr>
<td>(0.5,−0.25)</td>
<td>24.19</td>
<td>84.46</td>
<td>32.24</td>
<td>22.18</td>
<td>32.01</td>
<td>15.63</td>
<td>15.80</td>
</tr>
<tr>
<td>(0.5,0.5)</td>
<td>30.28</td>
<td>111.61</td>
<td>45.25</td>
<td>30.40</td>
<td>43.07</td>
<td>22.51</td>
<td>22.40</td>
</tr>
<tr>
<td>(0.75,0.25)</td>
<td>26.85</td>
<td>89.77</td>
<td>40.10</td>
<td>25.24</td>
<td>36.18</td>
<td>18.31</td>
<td>18.66</td>
</tr>
<tr>
<td>(1,−0.25)</td>
<td>15.26</td>
<td>41.15</td>
<td>17.45</td>
<td>11.86</td>
<td>16.50</td>
<td>9.07</td>
<td>8.61</td>
</tr>
<tr>
<td>(0.25,−1.5)</td>
<td>8.80</td>
<td>19.44</td>
<td>8.44</td>
<td>6.18</td>
<td>8.24</td>
<td>4.81</td>
<td>4.60</td>
</tr>
<tr>
<td>(1.5,−0.25)</td>
<td>13.59</td>
<td>33.18</td>
<td>14.93</td>
<td>10.42</td>
<td>14.17</td>
<td>7.72</td>
<td>7.75</td>
</tr>
<tr>
<td>(0.75,−0.75)</td>
<td>11.75</td>
<td>28.97</td>
<td>12.52</td>
<td>8.91</td>
<td>11.97</td>
<td>6.82</td>
<td>6.58</td>
</tr>
<tr>
<td>(1,1)</td>
<td>24.74</td>
<td>83.47</td>
<td>35.59</td>
<td>23.90</td>
<td>32.53</td>
<td>17.58</td>
<td>17.01</td>
</tr>
<tr>
<td>(1.5,−1.5)</td>
<td>3.00</td>
<td>4.20</td>
<td>2.31</td>
<td>1.92</td>
<td>2.22</td>
<td>1.61</td>
<td>1.59</td>
</tr>
<tr>
<td>(2,2)</td>
<td>14.13</td>
<td>36.63</td>
<td>15.48</td>
<td>11.23</td>
<td>15.17</td>
<td>8.35</td>
<td>8.11</td>
</tr>
</tbody>
</table>

**Notes:**
1. $(b_1, b_2)$ indicates the amount of process shift in terms of standard deviations, e.g. (0.5, 0.5) indicates that both the means of the two processes have shifted by 0.5 standard deviation.
2. The boldfaced numbers indicate that their ARLs are better than those of the original ones.
When two quality characteristics are moderately correlated or highly correlated, Tables 7 and 8 show that, regardless of the process mean shifts, Rule 3 performs much better than the original one, while Rules 1, 2, and 4 are worst than the original one for most of the time. Overall, as shown in Tables 4, 5, 6, 7 and 8, Rule 3 is considered to be the most appropriate monitoring rule for the multivariate pre-control charts.

The above simulation results of comparison under the various pre-control conditions as well as the three different correlation coefficients showed that most of the time the monitoring Rule 3 had a better performance. Therefore, Rule 3 is recommended as the new monitoring rule for multivariate pre-control charts. The producer may also adjust the pre-control conditions based on their needs. The performances of ARL1s under the various Type I errors/pre-control conditions and three different process mean shifts are summarized in Table 9.

### 3.3. A comparison of the performances of multivariate pre-control and Hotelling $T^2$ control charts

The Hotelling $T^2$ control chart is based on the statistics proposed by Hotelling(1947), where $T^2$ follows a chi-square distribution with degree of freedom $p$. It is an ellipse with probability $1-\alpha$. Assume that the parameters of the population, $\mu$ and $\Sigma$, are known, one can estimate $T^2$ by the sample points during the process monitoring. $T^2 > F_2(\alpha)$ indicates that the process is out-of-control, otherwise, the process is in-control. The sample size of the Hotelling $T^2$ control chart is set to $n = 3$ since the sample size of monitoring Rule 3 of the multivariate pre-control chart is also set to $n = 3$.

Assume that the process data $\mu$ and $\Sigma$ follows a bivariate normal distribution, where $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the covariance matrix $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. In order to compare the performances of ARLs (Average Run Lengths) of both charts, we set both in-control ARLs to have the same length $1/(1-\alpha)$, and then compare their out-of-control ARL1s, where $\text{ARL1} = 1/(1-\beta)$. A shorter ARL1 indicates a better performance of detection. If one sets the Green zone of a bivariate pre-control chart to be 0.86 of the process zone, the yellow zone to be between 0.86 and 0.999 of the process zone, and red zone to be beyond 0.999 of the process zone, then the simulation results of a comparison of ARL1s for both the bivariate pre-control and Hotelling $T^2$ charts under different correlation coefficients are summarized in Table 10.

Table 10 indicates that with the same fixed in-control ARLs, the Hotelling $T^2$ control chart has shorter ARL1s. In other words, the Hotelling $T^2$ control chart performs better than the bivariate...
pre-control chart in terms of the speed of detection when the process is out-of-control. However, if the producer focuses on assessing the short-term process capability during the set-up stage as well as using it as an easy and convenient monitoring tool for practitioners, then a bivariate/multivariate pre-control chart is recommended. On the other hand, if the producer has a sophisticated automatic system for data collection and analysis, then the Hotelling $T^2$ control chart is suggested since it has a better out-of-control ARL.

Combining the merits of both, one can also consider to use the multivariate control chart to assess the short-term process capability during the set up stage, and then use the Hotelling $T^2$ control chart to continuously monitor the process during the mass production stage.

### 4. Numerical example

A plastic manufacturer located in Taiwan produces glass fiber, insulation paper and epoxy resin. Among the various manufacturing processes, the tensile and impulse strengths of a PBT product are two important correlated characteristics that need to be controlled. After completing the define, measure, analysis and improve stages of the Six Sigma program, the factory manager has decided to use a multivariate pre-control chart to detect the process changes and prevent defects from happening at the control stage. The following procedures are suggested in performing a multivariate process control for practitioners.

**Step 1. Decide on the product’s key quality characteristics and their specifications.**

In this case, the product’s quality characteristics are the correlated tensile and impulse strengths of a PBT product and the specifications of these quality characteristics are 65% and 9 kg cm/cm, respectively.

**Step 2. Decide on the set-up rule for the multivariate pre-control chart.**

In this case, we suggest to use sample size of 10 for the new set-up rule, then the confidence level will be closed to 0.8 if all the 10 samples fall in the green zone (please refer to Section 3.1 for details).

**Step 3. Decide on the monitoring rule for the multivariate pre-control chart.**

Instead of using the IX-MR (individual X and moving range) for each characteristic,
we propose the use of Rule #3 with sample size of 3 as the new monitoring rule, i.e., for the cases of 3 green, 2 green 1 yellow, the process is considered stable. If 1 green 2 yellow occurs, then two more samples need to be added.

5. Conclusion

The quality of the output of a production process is often measured by the joint level of several correlated characteristics. Through multivariate control charts, one will be able to detect process changes and prevent defects from occurring by identifying and eliminating assignable causes of variation. In contrast to the traditional control charts, pre-control charts focus on evaluating the process capability during the set-up stage and detecting the process change during the mass production stage. It is a simple and time-saving tool for the monitoring of process performance. A proper determination of the sample size for the multivariate normal pre-control chart as well as its set-up and monitoring rules during the set-up and mass-production stages have been thoroughly discussed in this paper. The findings and contributions of this paper can be summarized as follow:

(1) It is suggested that the sample size be increased to 10 for preventing a false-judgment from happening during the set-up stage.

(2) We propose the new monitoring rule (Rule #3 with sample size $n = 3$) for the multivariate pre-control chart during the mass-production stage. The performances of ARL1's under the various Type I errors/pre-control conditions and three different process shifts are summarized in Table 9, which can be used as a reference for different industries.

(3) We compare the performances of the multivariate pre-control and Hotelling $T^2$ control charts and then suggest that the merits of both charts be combined during the set-up and mass-production stages.

References


