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New Capability Indices for Evaluating the Performance of Multivariate Manufacturing Processes

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Generally, an industrial product has more than one quality characteristic. In order to establish performance measures for evaluating the capability of a multivariate manufacturing process, several multivariate process capability indices have been developed in the past few years. Among them, Taam's MC_p and MC_{pm} indices have the drawback of overestimation and Hubele's three-component capability vector lacks simplicity in practice. In this article, taking the correlation among multiple quality characteristics into account, we develop two novel indices; NMC_p and NMC_{pm}. Using two numerical examples we demonstrate that the true performance of multivariate processes are accurately reflected in our NMC_p and NMC_{pm} indices and in their associated interval estimates. Finally, simulation results show that our indices outperform both those of Taam and Hubele. Copyright © 2009 John Wiley & Sons, Ltd.

Keywords: multivariate manufacturing process; multivariate process capability indices; performance evaluation

1. Introduction

Good quality products are the key to success in business. With the advent of modern technology, manufacturing processes have become very sophisticated; a single quality characteristic can no longer reflect a product's quality. For example, stencil printing is one of the most cost-effective processes for solder paste deposition and it has been widely used in traditional high-volume surface mount assembly (Pan *et al.*[1]). The need for higher pin count, higher performance, smaller size and lighter weight has driven the development of advanced packaging such as quad flat package (QFP), ball grid array, chip scale package and flip chip. In the solder paste stencil printing process, solder deposited volume, area and height are the three quality characteristics and there is a high correlation among them.

To establish performance measures for evaluating the capability of a multivariate manufacturing process, several multivariate process capability indices have been developed based on the further extensions from the univariate domain Unfortunately, the engineering tolerance region in a multivariate case is not simply represented by end points on a number line.The actual process region is not a straightforward function of the process standard deviation (Chan *et al.*[2]). Thus, one of the major problems in developing multivariate process capability indices is to establish an engineering tolerance region and a process region. It has been argued that the three-component capability vector proposed by Hubele *et al.*[3] lacks simplicity and could be confusing in its interpretation and use. On the other hand, the indices proposed by Taam *et al.*[4] do not take into account the correlation between multiple quality characteristics. In this article, we seek to resolve both these issues.

2. Literature review

In the past univariate process capability indices have been used to measure the process performance. Various multivariate statistical methods are now employed when quality characteristics are interdependent or correlated. Wang and Chen[5] simplified the computation of multivariate process capability by using principal component analysis. Chan *et al.*[2] proposed a multivariate process capability index C_{pm} using the concept of Mahalanobis distance. Chen[6] proposed a general multivariate capability index that allows

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elliptical and rectangular specifications. Foster *et al.*[7] later proposed a new multivariate capability index using a process-oriented basis representation.

Viewing the multivariate process capability indices as an extension of the univariate concept, Hubele *et al.*[3] proposed a composite measure for process capability based on two quality characteristics. Their three-component capability vector is defined as $[C_{pM}, PV, LI]$, where the first component of the capability vector, C_{pM} as shown in Equation (1), is a ratio of areas or volumes analogous to the ratio of lengths of the univariate C_p index. The numerator of Equation (1) is the volume of an engineering tolerance region, whereas the denominator is the volume of a modified process region

$$C_{pM} = \left(\sum_{i=1}^{v} (USL_i - LSL_i) \Big/ \sum_{i=1}^{v} (UPL_i - LPL_i) \right)^{1/v} \tag{1}$$

where USL_i is the upper specification limit for the ith quality characteristic, LSL_i is the lower specification limit for the ith quality characteristic, UPL_i and LPL_i are the upper and lower specification limits of a modified process region for the ith quality characteristic, respectively and v is the number of quality characteristics. The second component PV of the capability vector, as shown in Equation (2), measures the closeness of the process mean to the target.

$$PV = Pr\left[T^2 > \frac{v(n-1)}{n-v} F_{(v,n-v)} \right] \tag{2}$$

where $T^2 = n(\bar{\mathbf{X}} - \mathbf{T})' \mathbf{S}^{-1} (\bar{\mathbf{X}} - \mathbf{T})$, $\mathbf{T}$ is the target vector, $\bar{\mathbf{X}}$ is the sample mean vector, $\mathbf{S}$ is the sample covariance matrix, n is the sample size and $F_{(v,n-v)}$ is the F distribution with v and $n-v$ degrees of freedom. The third component of the capability vector LI compares the modified process region with the engineering tolerance region. It indicates whether the modified process region falls outside the engineering tolerance region. When the entire modified process region falls within the engineering tolerance region, LI is equal to 1. Otherwise, 0 is assigned to LI.

Taam *et al.*[4] proposed two multivariate process indices MC_p and MC_{pm}. Their multivariate process capability index MC_{pm} is defined as the ratio of two volumes, i.e.

$$MC_{pm} = \frac{vol.(R_1)}{vol.(R_2)} \tag{3}$$

where R_1 is a modified engineering tolerance region (see Figure 1) and R_2 is a scaled 99.73% process region, which is an elliptical region if the underlying process distribution is assumed to be multivariate normal. Moreover, the modified engineering tolerance region is the largest ellipsoid that is centered at the target and falls within the original engineering tolerance region. Thus, the MC_{pm} index can be rewritten as

$$MC_{pm} = MC_p \frac{1}{D} \tag{4}$$

where $D = (1 + (\boldsymbol{\mu} - \mathbf{T})' \boldsymbol{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{T}))^{1/2}$ is a correcting factor if the process mean is deviated from the target and the MC_p index represents the ratio of a modified tolerance region with respect to the process variability as written in the following equation:

$$MC_p = \frac{\left(\prod_{i=q}^{v} r_i \right) \pi^{v/2} [\Gamma(v/2) + 1]^{-1}}{|\boldsymbol{\Sigma}|^{1/2} (\pi K(v))^{v/2} [\Gamma(v/2) + 1]^{-1}} \tag{5}$$

where $r_i = (USL_i - LSL_i)/2$, $i = 1, \ldots, v$, $|\cdot|$ is a notation of determinant and $\Gamma(\cdot)$ is a Gamma function.

Wang *et al.*[8] reviewed the three multivariate process capability indices proposed by Hubele *et al.*[3], Taam *et al.*[4] and Chen[6]. They pointed out that Hubele's three-component capability vector lacks simplicity and could be confusing in its interpretation and use. Although Taam's MC_{pm} index accurately reflects process variability and centeredness, it does not take into account the correlation between multiple quality characteristics. We seek to resolve these issues. In Section 2, we show that it would be more appropriate to revise Taam's modified engineering tolerance region by considering the correlation of multiple quality characteristics.

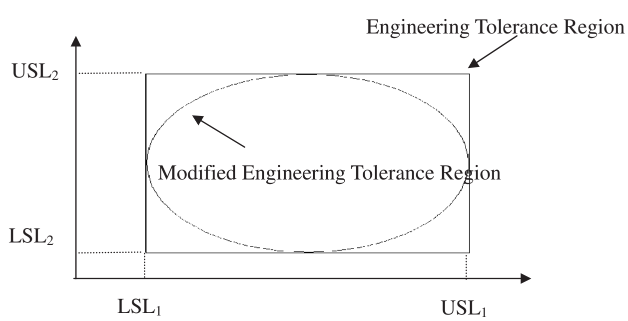

Figure 1. Illustration of engineering tolerance region and modified engineering tolerance region

3. Modification of the existing multivariate process indices

Assuming the manufacturing process follows a multivariate normal distribution, the MC_p index will be equal to 1 if the manufacturing process region falls completely within an engineering tolerance region. In Appendix A, we prove that the calculation of the MC_p index proposed by Taam *et al.*[4] can be simplified to

$$MC_p = |\boldsymbol{\rho}|^{-1/2} \tag{6}$$

where $\boldsymbol{\rho}$ is a correlation matrix. Equation (6) implies that the value of MC_p index may be greater than 1 since the value of the determinant of a correlation matrix is between 0 and 1. In other words, the value of MC_p index will be greater than 1 if the quality characteristics are not independent, which causes an overestimation of the true process performance. Similarly, the $MC_{pm} = MC_p/D$ index proposed by Taam *et al.*[4] has the same drawback as the MC_p index when the multiple quality characteristics are not independent. Thus, we revise Taam's modified engineering tolerance region based on the assumption that the correlation of multiple quality characteristics is consistent with the correlation among specifications. The relationship between Taam's modified engineering tolerance region (the regular one) and our revised engineering tolerance region (the slant one) for a process with a bivariate quality characteristic is illustrated in Figure 2.

To overcome the drawback of overestimation using the MC_p and MC_{pm} indices, a revised engineering tolerance region is proposed as below

$$E_{d,\mathbf{A}^*,\mathbf{T}} = \{\mathbf{X} \in \mathbf{R}^v | (\mathbf{X}-\mathbf{T})'(\mathbf{A}^*)^{-1}(\mathbf{X}-\mathbf{T}) = d^2\} \tag{7}$$

where the elements of matrix $\mathbf{A}^*$ are given by

$$\rho_{ij}\left(\frac{USL_i - LSL_i}{2d}\right)\left(\frac{USL_j - LSL_j}{2d}\right), \quad i,j = 1,\ldots,v \tag{8}$$

where $\mathbf{T}$ is the target vector, ρ_{ij} represents the correlation coefficient between the ith and jth quality characteristics and $(USL_i - LSL_i)$ denotes the ith specification width for each side of the rectangle circumscribed to the ellipsoid $E_{d,\mathbf{A}^*,\mathbf{T}}$. Similar to the MC_{pm} ratio of the two volumes shown in Equation (3), our proposed multivariate process capability index can be defined as

$$NMC_{pm} = \frac{Vol.(E_{d,\mathbf{A}^*,\mathbf{T}})}{Vol.(E_{d,\boldsymbol{\Sigma},\boldsymbol{\mu}})}(1+(\boldsymbol{\mu}-\mathbf{T})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{T}))^{-1/2} \tag{9}$$

If $d^2 = \chi^2_{1-\alpha,v}$, then the NMC_{pm} index can be written as

$$NMC_{pm} = \frac{|\mathbf{A}^*|^{1/2}(\pi\chi^2_{1-\alpha,v})^{v/2}\left[\Gamma\left(\frac{v}{2}+1\right)\right]^{-1}}{|\boldsymbol{\Sigma}|^{1/2}(\pi\chi^2_{1-\alpha,v})^{v/2}\left[\Gamma\left(\frac{v}{2}+1\right)\right]^{-1}}(1+(\boldsymbol{\mu}-\mathbf{T})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{T}))^{-1/2} = NMC_p/D \tag{10}$$

where $NMC_p = (|\mathbf{A}^*|/|\boldsymbol{\Sigma}|)^{1/2}$ and $D = (1+(\boldsymbol{\mu}-\mathbf{T})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{T}))^{1/2}$. The term D in Equation (10) denotes a function of Mahalanobis distance between the process mean and target vector $\mathbf{T}$. It can be used to measure the process deviation from target vector $\mathbf{T}$. Note that C_p and C_{pm} can be considered as a special case of NMC_p and NMC_{pm} if $v=1$ and $1-\alpha = 0.9973$. Thus, the NMC_p index can be used to evaluate the performance of process precision (i.e. the variability in relationship to the revised engineering tolerance region) and the NMC_{pm} index can be used to evaluate both process precision and accuracy (i.e. the deviation from the target). Given a random sample of n measurements with v quality characteristics (i.e. $\mathbf{X}_1,\ldots,\mathbf{X}_n$) from a multivariate manufacturing process, an estimator for our NMC_p index can be written as

$$\widehat{NMC}_p = \left(\frac{|\mathbf{A}^*|}{|\mathbf{S}|}\right)^{1/2} \tag{11}$$

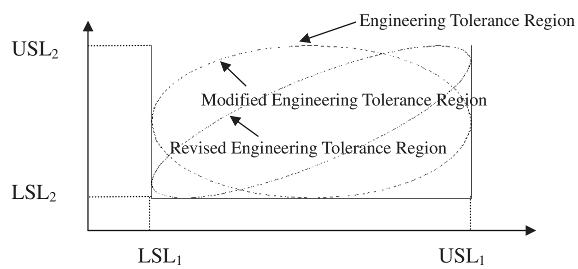

Figure 2. Relationship between Taam's modified region and our revised engineering tolerance region

where $\mathbf{S}=(n-1)^{-1}\sum_{i=1}^{n}(\mathbf{X}_i-\bar{\mathbf{X}})(\mathbf{X}_i-\bar{\mathbf{X}})'$ and $\bar{\mathbf{X}}=n^{-1}\sum_{i=1}^{n}\mathbf{X}_i$ represent the sample covariance matrix and the sample mean vector, respectively. Furthermore, the rth moment of $\widehat{NMC}_p$ can be derived as

$$\frac{1}{b_r}(NMC_p)^r \tag{12}$$

where

$$b_r=\left(\frac{2}{n-1}\right)^{rv/2}\prod_{i=1}^{v}\left(\frac{\Gamma((n-i)/2)}{\Gamma((n-i-r)/2)}\right)$$

(see Appendix B for details). Then, the expected value and variance of $\widehat{NMC}_{pm}$ can be written as

$$E(\widehat{NMC}_p)=\frac{1}{b_1}NMC_p$$

$$Var(\widehat{NMC}_p)=\left(\frac{1}{b_2}-\frac{1}{b_1^2}\right)(NMC_p)^2$$

Note that $\widehat{NMC}_p$ is a biased estimator and $b_1\widehat{NMC}_p$ is an unbiased estimator of the NMC_p index. Similarly, the estimator of the NMC_p index can be written as

$$\widehat{NMC}_{pm}=\left(\frac{|\mathbf{A}^*|}{|\mathbf{S}^*|}\right)^{1/2}$$

where $\mathbf{S}^*=(n-1)^{-1}\sum_{i=1}^{n}(\mathbf{X}_i-\mathbf{T})(\mathbf{X}_i-\mathbf{T})'$. Furthermore, the rth moment of $\widehat{NMC}_{pm}$ can be derived as

$$\frac{1}{b_r^*}(NMC_{pm})^r$$

where

$$b_r^*=\left(\frac{2^{v-1}}{(n-1)^v(1+\lambda/n)}\right)^{r/2}e^{\lambda/2}\left(\sum_{j=0}^{\infty}\frac{(\lambda/2)^j\Gamma((n+2j-r)/2)}{j!\,\Gamma((n+2j)/2)}\right)^{-1}\prod_{i=1}^{v-1}\left(\frac{\Gamma((n-i)/2)}{\Gamma((n-i-r)/2)}\right)$$

(see Appendix B for details). Then, the expected value and variance of $\widehat{NMC}_{pm}$ can be written as:

$$E(\widehat{NMC}_{pm})=\frac{1}{b_1^*}NMC_{pm}$$

$$Var(\widehat{NMC}_{pm})=\left(\frac{1}{b_2^*}-\frac{1}{(b_1^*)^2}\right)(NMC_{pm})^2$$

Apparently, $\widehat{NMC}_{pm}$ is a biased estimator and $b_1^*\widehat{NMC}_{pm}$ is an unbiased estimator of the NMC_{pm} index.

Based on the sampling distributions of $\widehat{NMC}_p$ and $\widehat{NMC}_{pm}$, we further prove that the $100(1-\alpha)\%$ confidence interval for the NMC_p index is

$$\left[\widehat{NMC}_p\sqrt{w_{\alpha/2}},\ \widehat{NMC}_p\sqrt{w_{1-\alpha/2}}\right] \tag{13}$$

and the $100(1-\alpha)\%$ confidence interval for the NMC_{pm} index is

$$\left[\widehat{NMC}_{pm}\sqrt{\frac{w^*_{\alpha/2}}{1+\lambda/n}},\ \widehat{NMC}_{pm}\sqrt{\frac{w^*_{1-\alpha/2}}{1+\lambda/n}}\right]$$

where w_α and w_α^* are the α percentiles of $\prod_{i=1}^{v}\chi^2_{n-i}/(n-1)^v$ and $\chi^2_n(\lambda)\prod_{i=1}^{v-1}\chi^2_{n-i}/(n-1)^v$ distributions, respectively (see Appendix C for details). Note that χ^2_{n-i}, $i=1,\ldots,v$, are independent Chi-square distributions with $(n-i)$ degrees of freedom, $\chi^2_n(\lambda)$ is a non-central Chi-square distribution with n degrees of freedom and the non-centrality parameter $\lambda=n(\boldsymbol{\mu}-\mathbf{T})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{T})$. Since λ is usually unknown and it can be estimated by $\hat{\lambda}=n(\bar{\mathbf{X}}-\mathbf{T})'\mathbf{S}^{-1}(\bar{\mathbf{X}}-\mathbf{T})$, an approximate $100(1-\alpha)\%$ confidence interval for the NMC_{pm} index can be written as

$$\left[\widehat{NMC}_{pm}\sqrt{\frac{w^*_{\alpha/2}}{1+\hat{\lambda}/n}},\ \widehat{NMC}_{pm}\sqrt{\frac{w^*_{1-\alpha/2}}{1+\hat{\lambda}/n}}\right] \tag{14}$$

The above interval estimation for a multivariate capability index provides a range of plausible values for an estimate that accounts for sampling error.

4. Comparison of various multivariate process capability indices

Our simulation study compares the performance of our two multivariate process capability indices with the three multivariate process capability indices proposed by Chan *et al.*[2], Hubele *et al.*[3] and Taam *et al.*[4]. The specifications and the target values for a hypothetical bivariate process are listed in Table I.

According to Equation (8), the matrix $\mathbf{A}^*$ for process I is given by

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

where ρ is the correlation coefficient between two quality characteristics $\mathbf{X}_1$ and $\mathbf{X}_2$. Assuming that the hypothetical bivariate process I follows a multivariate normal distribution with mean vector $\boldsymbol{\mu}'=[0\ 0]$ and covariance matrix $\boldsymbol{\Sigma}=\mathbf{A}^*$, a 99.73% process region $E_{\chi^2_{0.9973,2},\boldsymbol{\Sigma},\boldsymbol{\mu}}=\{\mathbf{X}\in\mathbf{R}^2|(\mathbf{X}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})=\chi^2_{0.9973,2}\}$ and a 99.73% revised engineering tolerance region $E_{\chi^2_{0.9973,2},\mathbf{A}^*,\mathbf{T}}=\{\mathbf{X}\in\mathbf{R}^2|(\mathbf{X}-\mathbf{T})'(\mathbf{A}^*)^{-1}(\mathbf{X}-\mathbf{T})=\chi^2_{0.9973,2}\}$ are equivalent. Thus, the value of multivariate process capability index should be nearly equal to 1 when the process mean is equal to the target. After conducting 10 000 times simulation, the computation results of various multivariate process capability indices for different combinations of sample size (n) and correlation coefficient (ρ) are summarized in Table II. Note that we assume that Chan's tolerance ellipsoid is equivalent to the modified engineering tolerance region used in the MC_{pm} index since Chan *et al.*[2] did not show how to convert structure matrix from an engineering tolerance region.

Apparently, it is unreasonable to use Chan's $\hat{\underset{\sim}{C}}_{pm}$ index as its values are significantly greater than 1. The simulation results further reveal that the multivariate process capability is overestimated by Taam's $\widehat{MC}_p$ and $\widehat{MC}_{pm}$ indices since their indices increase as the correlation of quality characteristic increases. In contrast, the two novel $\widehat{NMC}_p$ and $\widehat{NMC}_{pm}$ indices fall within the neighborhood of 1 for different correlation coefficients when the sample size (n) is greater than 30. Therefore, our proposed $\widehat{NMC}_p$ and $\widehat{NMC}_{pm}$ indices are more appropriate than Taam's $\widehat{MC}_p$ and $\widehat{MC}_{pm}$ indices since they are robust to the change in the correlation coefficients. Furthermore, all the values of the first and second components of Hubele's index are close to 1 and 0.5 for different correlation coefficients. These simulation results indicate that the first and second components ($\hat{C}_{pM}$ and PV) of Hubele's index are appropriate to evaluate the performance of a multivariate manufacturing process. According to Equation (A2), the lower and upper process limits of a modified process region proposed by Hubele *et al.*[3] can be expressed as $UPL_i=\mu_i+\sqrt{\sigma_i^2\chi^2_{1-\alpha,v}}$ and $LPL_i=\mu_i-\sqrt{\sigma_i^2\chi^2_{1-\alpha,v}}$, where σ_i^2, $i=1,\ldots,v$, is the variance of ith quality characteristics. Thus, we can further show that the relationship between the C_{pM} index and the NMC_p

Table I. The specifications and target values for a hypothetical bivariate process I

Quality characteristic	Target	USL	LSL
X_1	0	3.4393	−3.4393
X_2	0	3.4393	−3.4393

Table II. The simulation results of various capability indices for different combinations of sample size (n) and correlation coefficient (ρ)

		Hubele's index			Chan's index	Taam's indices		Our proposed indices	
ρ	n	$\hat{C}_{pM}$	PV	LI	$\hat{\underset{\sim}{C}}_{pm}$	$\widehat{MC}_p$	$\widehat{MC}_{pm}$	$\widehat{NMC}_p$	$\widehat{NMC}_{pm}$
0.1	30	1.0227	0.4957	0.2171	3.4846	1.0798	1.0413	1.0556	1.0180
	50	1.0132	0.5023	0.2095	3.4665	1.0481	1.0263	1.0321	1.0107
	100	1.0070	0.4927	0.2031	3.4541	1.0273	1.0166	1.0169	1.0062
0.5	30	1.0226	0.5021	0.2514	3.4927	1.2406	1.1964	1.0576	1.0200
	50	1.0123	0.5039	0.2354	3.4677	1.2015	1.1767	1.0314	1.0101
	100	1.0065	0.4965	0.2399	3.4546	1.1784	1.1661	1.0164	1.0058
0.9	30	1.0272	0.5007	0.3452	3.5224	2.4641	2.3759	1.0733	1.0348
	50	1.0141	0.4966	0.3317	3.4825	2.3945	2.3440	1.0384	1.0165
	100	1.0068	0.4962	0.3245	3.4607	2.3392	2.3151	1.0184	1.0079

index is given by

$$\begin{aligned} C_{pM} &= \left(\prod_{i=1}^{v} (USL_i - LSL_i) \Bigg/ \prod_{i=1}^{v} (UPL_i - LPL_i) \right)^{1/v} \\ &= \left(\prod_{i=1}^{v} (USL_i - LSL_i) \Bigg/ \left(2\sqrt{\chi^2_{1-\alpha,v}} \prod_{i=1}^{v} \sigma_i \right) \right)^{1/v} \\ &= (|\mathbf{A}^*| / |\mathbf{\Sigma}|)^{1/2v} \\ &= (NMC_p)^{1/v} \end{aligned} \tag{15}$$

Table III. The mean vectors and covariance matrices of four hypothetical bivariate processes

	Hypothetical bivariate processes			
	II	III	IV	V
Mean vector ($\boldsymbol{\mu}$)	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\mathbf{\Sigma}^{1/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\mathbf{\Sigma}^{1/2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
Covariance matrix ($\mathbf{\Sigma}$)	$\begin{bmatrix} 1 & \sqrt{2}\rho \\ \sqrt{2}\rho & 2 \end{bmatrix}$	$\begin{bmatrix} 2 & 2\rho \\ 2\rho & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$	$\begin{bmatrix} 2 & 2\rho \\ 2\rho & 2 \end{bmatrix}$

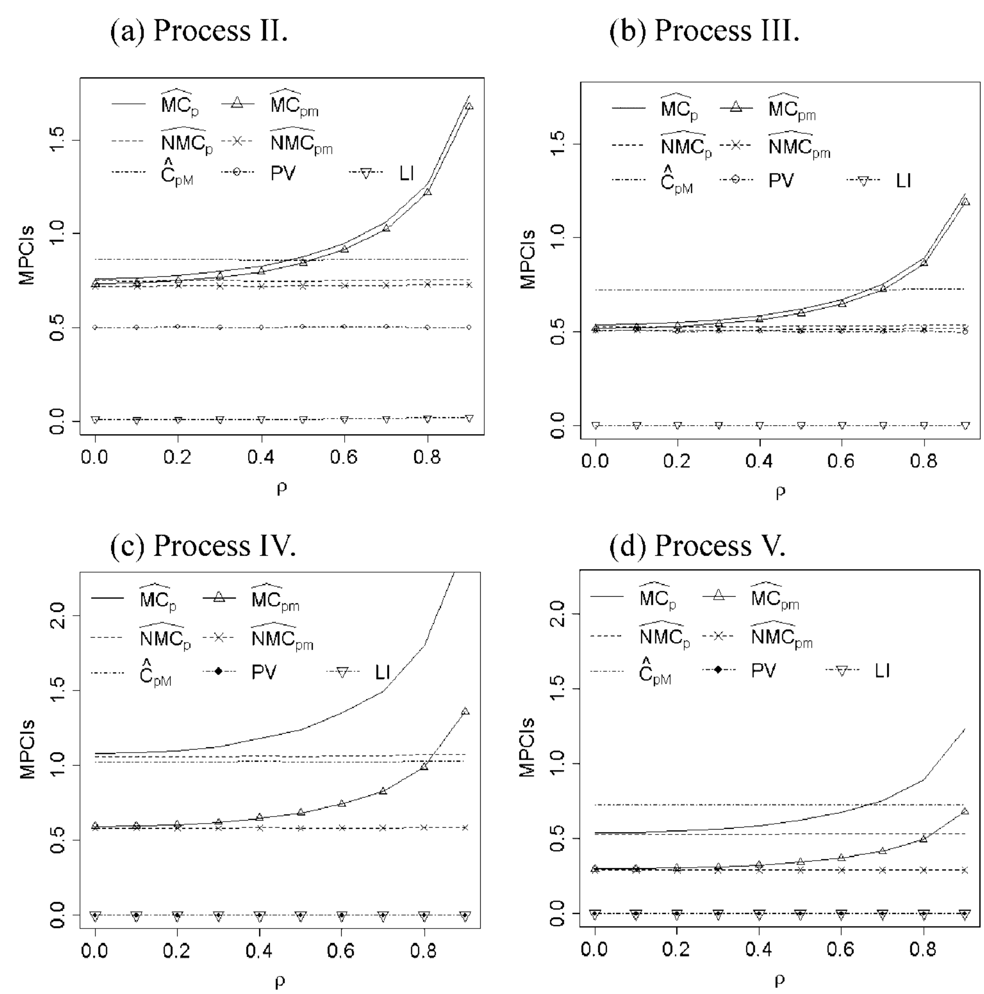

Figure 3. The simulation results of various multivariate process capability indices for different correlation coefficients (ρ) under processes II, III, IV and V

Although Hubele's C_{pM} index is based on the ratio of two volumes of rectangular regions and our NMC_p index is based on the ratio of two volumes of ellipsoids, it is found that both the C_{pM} and NMC_p indices can provide the same information relating to the process precision.

To compare the performance of multivariate process capability indices, we consider four hypothetical bivariate processes II, III, IV and V as listed in Table III. Note that the target values and specifications of four hypothetical processes are the same as those of process I (see Table I for details).

In Table III, process II represents the variation of one quality characteristic, which is greater than 1, process III represents the variations of two quality characteristics, which are greater than 1, process IV represents the process mean vector that is deviated from the target vector and process V represents both the variations of two quality characteristics, which are greater than 1 and where the process mean vector is deviated from the target vector, respectively. Moreover, Mahalanobis distances between the target vector and the process mean vector are equal to 2 for processes IV and V. After conducting 10 000 times simulation, the computation results of various multivariate process capability indices ($n=30$) for processes II, III, IV and V under different correlation coefficient (ρ) are illustrated in Figure 3(a–d), respectively. Figure 3(a–d) shows that the true capability of a multivariate manufacturing process cannot be correctly reflected by Taam's $\widehat{MC}_p$ and $\widehat{MC}_{pm}$ indices since the process capability will be overestimated as the correlation of quality characteristics increases. In contrast, all the values of our $\widehat{NMC}_p$ index in Figure 3(a–d) are either less than or close to 1 for different correlation coefficients and all the values of our $\widehat{NMC}_{pm}$ index in Figure 3(a–d) are less than 1, which indicate that the process precision and accuracy can be correctly reflected by the $\widehat{NMC}_p$ and $\widehat{NMC}_{pm}$ indices.

Therefore, our proposed NMC_p and NMC_{pm} indices are more appropriate than Taam's MC_p and MC_{pm} indices for evaluating the performance of multivariate manufacturing processes. Moreover, all the values of the three components of Hubele's index $[\hat{C}_{pM}, PV, LI]$ in Figure 3(a–d) are less than or close to 1, 0.5, 0 and 0, respectively. It indicates that Hubele's index is also suitable to evaluating the performance of multivariate manufacturing processes. Notice that all the values of the second component (PV) of Hubele's index in Figure 3(c) and (d) and the third component (LI) of Hubele's index in Figure 3(a–d) are close to 0.

5. Numerical examples

Example 1

Sultan[9] discussed an example in which the Brinell hardness (H) and tensile strength (S) are two quality characteristics of an industrial product. The engineering tolerances for H and S are given by (112.7, 241.3) and (32.7, 73.3), respectively, and the target vector of H and S is $\mathbf{T}'=[177, 53]$. After collecting 25 measurements as listed in Table IV, a multivariate process capability study is conducted (assuming that the process is in control).

By performing Shapiro–Wilk test, we found that the 25 collected measurements follow a multivariate normal distribution with the sample mean vector $\bar{\mathbf{X}}'=[177.2, 52.316]$ and the sample covariance matrix $\mathbf{S}$, where

$$\mathbf{S}=\begin{bmatrix} 338 & 88.8925 \\ 88.8925 & 33.6247 \end{bmatrix}$$

Then, the matrix $\mathbf{A}^*$ can be obtained as below

$$\mathbf{A}^*=\begin{bmatrix} 349.52131 & 92.01022 \\ 92.01022 & 34.83724 \end{bmatrix}$$

The actual relationship among the 99.73% process region, 99.73% revised engineering tolerance region and the engineering tolerance region is illustrated in Figure 4.

Apparently, the process mean is close to the target and the '99.73% process region' is approximately equal to the '99.73% revised engineering tolerance region'. The comparison results of using various multivariate process indices for estimating the performance of

Table IV. The 25 measurements of Brinell hardness (H) and tensile strength (S) for an industrial product

H	S	H	S	H	S
143	34.2	141	47.3	178	50.9
200	57.0	175	57.3	196	57.9
168	47.5	187	58.5	160	45.5
181	53.4	187	58.2	183	53.9
148	47.8	186	57.0	179	51.2
178	51.5	172	49.4	194	57.5
162	45.9	182	57.2	181	55.6
215	59.1	177	50.6		
161	48.4	204	55.1		

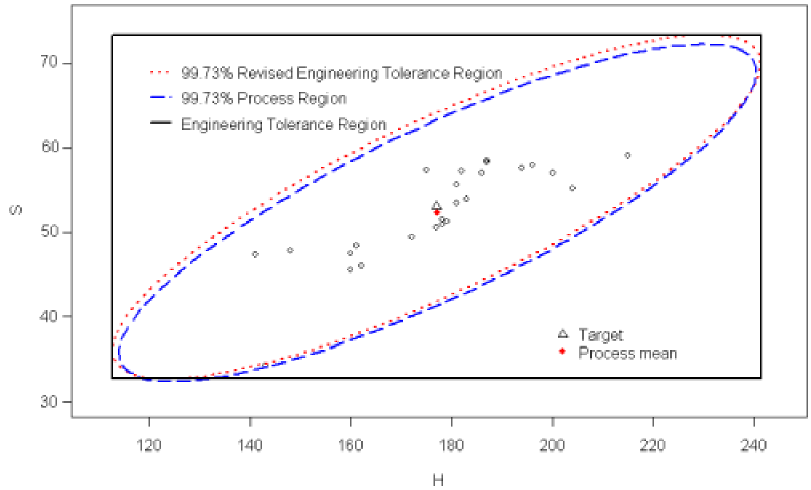

Figure 4. Relationship among 99.73% revised engineering tolerance region, 99.73% process region and engineering tolerance region in Example 1. This figure is available in colour online at www.interscience.wiley.com/journal/qre

Table V. Comparison of various multivariate process indices for Example 1

Hubele's index			Taam's indices		Our proposed indices	
$\hat{C}_{pM}$	PV	LI	$\widehat{MC}_p$	$\widehat{MC}_{pm}$	$\widehat{NMC}_p$	$\widehat{NMC}_{pm}$
1.02	0.54	1	1.88	1.83	1.04	1.01

Table VI. The specifications and target values for $QFP_{4\,\mathrm{mil},30}$ process

Quality characteristic	Target	USL	LSL
Deposited volume	0.0787	0.10250	0.0549
Deposited area	0.7870	0.96870	0.6052
Deposited height	0.1000	0.12765	0.07235

an industrial product are summarized in Table V. Since the estimated conforming rate for this example is 99.91% under the assumption of multivariate normality for the underlying process distribution and both our proposed indices $\widehat{NMC}_p = 1.04$ and $\widehat{NMC}_{pm} = 1.01$ are nearly equal to 1, which indicates that the 99.73% process region is close to the 99.73% revised engineering tolerance region and the process mean is close to the target (see Figure 4). By Equations (13) and (14), the 95% confidence intervals for the NMC_p and NMC_{pm} indices are [0.63, 1.44] and [0.63, 1.41], respectively. Thus, the true process performance can be correctly reflected by our proposed indices $\widehat{NMC}_p$ and $\widehat{NMC}_{pm}$, i.e. the process is capable. Whereas, the process capability is overestimated by Taam's two indices $\widehat{MC}_p = 1.88$ and $\widehat{MC}_{pm} = 1.83$ since the correlation among multiple quality characteristics is not taken into account. In this example, the process capability can also be reflected by Hubele's index since $\left[\hat{C}_{pM}, PV, LI\right] = [1.02, 0.54, 1]$ indicates that the modified process region is close to the engineering tolerance region, the process mean is near to the target value and the modified process region falls within the engineering tolerance region, respectively.

Example 2

The stencil used in the solder paste stencil printing process (see introduction) is laser-cut from stainless steel and there are two kinds of stencil thickness, 0.1 mm (4 mil) and 0.15 mm (6 mil). Both the stencils have the same pattern and there are five different aperture sizes, i.e. 30, 25, 20, 16 and 12 (unit: 1 mil = 0.0254 mm). In this example, we consider a $QFP_{4\,\mathrm{mil},30}$ process, where $QFP_{4\,\mathrm{mil},30}$ represents the stencil thickness as 4 mil and aperture size as 30. The specifications and target values for $QFP_{4\,\mathrm{mil},30}$ process are listed in Table VI.

Based on the 150 measurements provided by Pan *et al.*[1], we found that they follow a multivariate normal distribution with the sample mean vector $\bar{\mathbf{X}}' = [0.0786, 0.7871, 0.1000]$ and the sample covariance matrix $\mathbf{S}$, where

$$\mathbf{S} = \begin{bmatrix} 0.0000354 & 0.0001074 & 0.0000326 \\ 0.0001074 & 0.0020648 & -0.0000758 \\ 0.0000326 & -0.0000758 & 0.0000478 \end{bmatrix}$$

Table VII. Comparison of various multivariate process indices for $QFP_{4\,\text{mil},30}$ process

Hubele's index			Taam's indices		Our proposed indices	
$\hat{C}_{pM}$	PV	LI	$\widehat{MC}_p$	$\widehat{MC}_{pm}$	$\widehat{NMC}_p$	$\widehat{NMC}_{pm}$
1.10	0.00	1	26.39	24.60	1.20 [0.96, 1.42]	1.12 [0.66, 1.47]

Note: Values within brackets represent the interval estimates for our proposed indices.

Then, the matrix $\mathbf{A}^*$ for the revised engineering tolerance region can be obtained as below:

$$\mathbf{A}^* = \begin{bmatrix} 0.0000400 & 0.0001215 & 0.0000369 \\ 0.0001215 & 0.0023335 & -0.0000857 \\ 0.0000369 & -0.0000857 & 0.0000540 \end{bmatrix}$$

The comparison results of using various multivariate process indices for estimating the performance of $QFP_{4\,\text{mil},30}$ process are summarized in Table VII.

As the estimated conforming rate for this example is 99.98% under the assumption of multivariate normality and the indices $\widehat{NMC}_p = 1.20$ and $\widehat{NMC}_{pm} = 1.12$, one can conclude that the 99.73% process region is smaller than the 99.73% revised engineering tolerance region and the process mean is slightly deviated from the target (i.e. $\hat{D} = (1 + n(n-1)^{-1}(\bar{\mathbf{X}} - \mathbf{T})'\mathbf{S}^{-1}(\bar{\mathbf{X}} - \mathbf{T}))^{1/2} = 1.07$). Apparently, the true process performance can be reflected by our proposed indices $\widehat{NMC}_p$ and $\widehat{NMC}_{pm}$, i.e. the process is capable. Moreover, the sample correlation matrix $\mathbf{r}$ for this example is given by

$$\mathbf{r} = \begin{bmatrix} 1 & 0.3975 & 0.7934 \\ 0.3975 & 1 & -0.2414 \\ 0.7434 & -0.2414 & 1 \end{bmatrix}$$

Whereas, the process capability is overestimated by Taam's two indices, $\widehat{MC}_p = 26.39$ and $\widehat{MC}_{pm} = 24.60$. The consequence of this overestimation is caused by the high correlation among three multiple quality characteristics (see the above correlation matrix). In this example, the first component ($\hat{C}_{pM} = 1.10$) of Hubele's index indicates that the modified process region is smaller than the engineering tolerance region and the second component ($PV = 0$) of Hubele's index indicates that the process mean is far away from the target value. However, the second component (PV) of Hubele's index is defined as a p-value of the Hotelling T^2 statistic for the testing of $H_0: \boldsymbol{\mu} = \mathbf{T}$ versus $H_a: \boldsymbol{\mu} \neq \mathbf{T}$, where $\boldsymbol{\mu}$ and $\mathbf{T}$ are the process mean vector and the target vector, respectively. An important idea behind statistical testing and consequently the use of the p-value is that the reference distribution takes into account the sample size. In essence, we allow the process mean to be further away from the target when the sample size is small, but require it to be closer to the target when the sample size is large to obtain the same p-value. Thus, PV value of 0.00 in this example indicates that the process mean is statistically different from the target for such a large sample size (i.e. $n = 150$). As indicated by Wang *et al.*[8], the above interpretation for the second component (PV) of Hubele's index may not be easily understood by practitioners who do not have a solid statistical knowledge.

6. Conclusions and remarks

The distinction among the indices proposed by Hubele *et al.*[3], Taam *et al.*[4] and our proposed capability indices is: (1) the comparison regions and (2) the algebraic expression used to compute the indices. Hubele *et al.*[3] used multiple-dimensional rectangular regions to construct three components. The first component, analogous to C_p in higher dimensions, compares the volumes of the multiple-dimensional rectangular regions. The second component measures the distance between the process mean and the process target using the T^2 statistic and the third component compares the general location of the two multiple-dimensional rectangular regions. Using ellipsoid-shaped comparison regions (regular ones without taking the correlation among multiple quality characteristics into account), Taam's two indices, one analogous to C_p and the other analogous to C_{pm} from the univariate domain, are generated from the underlying multivariate normal distribution. Considering the correlation among multiple quality characteristics, we use the slant ellipsoid-shaped comparison region to develop the two novel capability indices. The simulation results show that our NMC_p and NMC_{pm} indices are more appropriate than Taam's MC_p and MC_{pm} indices since our indices are robust to the change in the correlation coefficients. Similar to Taam's MC_p and MC_{pm} indices, our proposed NMC_p and NMC_{pm} indices can also be viewed as an extension of the univariate process capability indices C_p and C_{pm}. Thus, the performance of process precision and accuracy for a multivariate manufacturing process can easily be understood by using our proposed multivariate process capability indices. Two numerical examples further demonstrate the usefulness of our proposed NMC_p and NMC_{pm} capability indices.

Moreover, it is worthy to note that (1) All the multivariate process capability indices including Taam's MC_p and MC_{pm}, Hubele's C_{pM} and PV indices as well as our proposed NMC_p and NMC_{pm} indices are based on the assumption of multivariate normality for the underlying distribution. Accordingly, any departure from this assumption could lead to erroneous results that include statistical properties, interval estimates and interpretation of process capability indices. Therefore, it is suggested that the multivariate normality assumption be checked by performing a statistical test, such as Shapiro–Wilk test prior to the process capability study. (2) Engineering tolerance zone or the intersection of the specifications would be a rectangular solid since the specifications for a product generally consist of a collection of individual specifications for each variable (see Jackson[10]). (3) Although Hubele's index is also appropriate in evaluating multivariate process capability, it only focuses on computing and interpreting the point estimates of the desired quantity. Thus, it is subject to statistical fluctuation. In contrast, our proposed NMC_p and NMC_{pm} process capability indices and their associated interval estimates, which may lead to sample size determination, can serve as a useful reference for quality practitioners.

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References

1. Pan J, Tonkay GL, Storer RH, Sallade RM, Leandri DJ. Critical variables of solder paste stencil printing for Micro-BGA and fine-pitch QFP. *IEEE Transactions on Electronics Packaging Manufacturing* 2004; **27**:125–132.
2. Chan LK, Cheng SW, Spiring FA. A multivariate measure of process capability. *International Journal of Modeling and Simulation* 1991; **11**:1–6.
3. Hubele NF, Shahriari H, Cheng CS. A bivariate process of capability vector. *Statistical Process Control in Manufacturing*, Keats JB, Montgomery DC (eds.). Marcel Dekker: New York, NY, 1991; 299–310.
4. Taam W, Subbaiah P, Liddy JW. A note on multivariate capability indices. *Journal of Applied Statistics* 1993; **20**:339–351.
5. Wang FK, Chen JC. Capability index using principal component analysis. *Quality Engineering* 1998; **11**:21–27.
6. Chen H. A multivariate process capability index over a rectangular solid tolerance zone. *Statistica Sinica* 1994; **4**:749–758.
7. Foster EJ, Barton RR, Gautam N, Truss LT, Tew JD. The process-oriented multivariate capability index. *International Journal of Production Research* 2005; **43**:2135–2148.
8. Wang FK, Hubele NF, Lawrence FP, Miskulin JD, Shahriari H. Comparison of three multivariate process capability indices. *Journal of Quality Technology* 2000; **32**:263–275.
9. Sultan TL. An acceptance chart for raw materials of two correlated properties. *Quality Assurance* 1986; **12**:70–72.
10. Jackson JE. Multivariate quality control: 40 years later. *Statistical Process Control in Manufacturing*, Keats JB, Montgomery DC (eds.). Marcel Dekker: New York, NY, 1991; 123–138.
11. Härdle W, Simar L. *Applied Multivariate Statistical Analysis*. Springer: Berlin, UA, 2003; 73–74.
12. Maman AD. Improved monitoring of multivariate process variability. *Journal of Quality Technology* 2005; **37**:32–39.
13. Dahel S, Giri N. Some distributions related to a noncentral wishart distribution. *Communications in Statistics—Theory and Methods* 1994; **23**:229–237.
14. Pearn WL, Kotz S, Johnson NL. Distributional and inferential properties of process capability indices. *Journal of Quality Technology* 1992; **24**:216–231.

Appendix A: Proof of an overestimation for Taam's MC_p and MC_{pm} indices

Suppose that $E_{d,\mathbf{A},\mathbf{X}_0}$ as shown in Equation (A1) is an ellipsoid with center $\mathbf{X}_0$ in $\mathbf{R}^v$

$$E_{d,\mathbf{A},\mathbf{X}_0} = \{\mathbf{X} \in \mathbf{R}^v | (\mathbf{X}-\mathbf{X}_0)'\mathbf{A}(\mathbf{X}-\mathbf{X}_0) = d^2\} \tag{A1}$$

where $\mathbf{A}$ is a positive-definite matrix and d is a constant. Then the rectangle circumscribed to the ellipsoid $E_{d,\mathbf{A},\mathbf{X}_0}$ can be defined by the following inequality:

$$x_{0i} - \sqrt{d^2 a^{ii}} \le x_i \le x_{0i} + \sqrt{d^2 a^{ii}}, \quad i = 1, \ldots, v \tag{A2}$$

where a^{ii} is the (i, i) element of $\mathbf{A}^{-1}$ and x_{0i} is the ith component of vector $\mathbf{X}_0$. The proof is given by Härdle and Simar[11].

Since the modified engineering tolerance region proposed by Taam *et al.*[4] is an ellipsoid inscribed to the original engineering tolerance region and parallel to the coordinate axes, it can be written as

$$E_{\chi^2_{1-\alpha,v}, \boldsymbol{\Sigma}_{Taam}, \mathbf{T}} = \{\mathbf{X} \in \mathbf{R}^v | (\mathbf{X}-\mathbf{T})'\boldsymbol{\Sigma}^{-1}_{Taam}(\mathbf{X}-\mathbf{T}) = \chi^2_{1-\alpha,v}\}$$

where $\mathbf{T}$ is the target vector, $\chi^2_{1-\alpha,v}$ is the $(1-\alpha)$th percentile of a Chi-square distribution with v degrees of freedom and $\boldsymbol{\Sigma}_{Taam}$ is a $v \times v$ diagonal matrix with $(USL_i - LSL_i)^2 \Big/ (2\sqrt{\chi^2_{1-\alpha,v}})^2$, $i = 1, \ldots, v$ elements. According to Equation (A2), it is easy to show that the sides of engineering tolerance region circumscribed to the ellipsoid $E_{\chi^2_{1-\alpha,v}, \boldsymbol{\Sigma}_{Taam}, \mathbf{T}}$ are $(USL_i - LSL_i)$, $i = 1, \ldots, v$. In other words, the rectangle circumscribed to the ellipsoid $E_{\chi^2_{1-\alpha,v}, \boldsymbol{\Sigma}_{Taam}, \mathbf{T}}$ is equal to the engineering tolerance region. Thus, the volume of $E_{\chi^2_{1-\alpha,v}, \boldsymbol{\Sigma}_{Taam}, \mathbf{T}}$ can be written as

$$|\boldsymbol{\Sigma}_{Taam}|^{1/2} (\pi \chi^2_{1-\alpha,v})^{v/2} \left[\Gamma\left(\frac{v}{2}+1\right)\right]^{-1} = \pi^{v/2} \left(\prod_{i=1}^{v} \frac{(USL_i - LSL_i)}{2}\right) \left[\Gamma\left(\frac{v}{2}+1\right)\right]^{-1} \tag{A3}$$

where $|\cdot|$ is a notation of determinant and $\Gamma(\cdot)$ is a Gamma function. Suppose that the process measurements follow a multivariate normal distribution with the mean vector $\mathbf{T}$ and a covariance matrix $\boldsymbol{\Sigma}$, which is given by

$$\rho_{ij}\left(\frac{USL_i-LSL_i}{2\sqrt{\chi^2_{1-\alpha,v}}}\right)\left(\frac{USL_j-LSL_j}{2\sqrt{\chi^2_{1-\alpha,v}}}\right),\quad i,j=1,\ldots,v$$

where ρ_{ij} represents the correlation coefficient between ith and jth quality characteristics. According to Equation (A2), it is easy to show that the manufacturing process region falls within the engineering tolerance region and the scaled $(1-\alpha)$% manufacturing process region is given by

$$E_{\chi^2_{1-\alpha,v},\boldsymbol{\Sigma},\mathbf{T}}=\{\mathbf{X}\in\mathbf{R}^v|(\mathbf{X}-\mathbf{T})'\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{T})=\chi^2_{1-\alpha,v}\}$$

Then, the volume of a scaled $(1-\alpha)$% manufacturing process region equals

$$\pi^{v/2}\left(\prod_{i=1}^{v}\frac{(USL_i-LSL_i)}{2}\right)\left[\Gamma\left(\frac{v}{2}+1\right)\right]^{-1}|\boldsymbol{\rho}|^{1/2} \tag{A4}$$

where $\boldsymbol{\rho}$ is a correlation matrix with unit diagonal elements and non-diagonal elements ρ_{ij}. According to Equations (A3) and (A4), the calculation of MC_p index can be simplified by

$$MC_p=|\boldsymbol{\rho}|^{-1/2} \tag{A5}$$

Equation (A5) indicates that the value of MC_p index might be greater than 1 (an overestimation occurs) since the value of the determinant of a correlation matrix is between 0 and 1 (when multiple quality characteristics are independent).

Appendix B: Derivation of the rth moments of $\widehat{NMC}_p$ and $\widehat{NMC}_{pm}$

Based on the definition of $\widehat{NMC}_p$, the rth moment of $\widehat{NMC}_p$ can be written as

$$E(\widehat{NMC}_p)^r=|\mathbf{A}^*|^{r/2}E(|\mathbf{S}|^{-r/2})$$

According to Maman[12], one can obtain

$$E(|\mathbf{S}|)^{-r/2}=\left(\frac{|\boldsymbol{\Sigma}|}{(n-1)^v}\right)^{-r/2}\prod_{i=1}^{v}E(\chi^2_{n-i})^{-r/2}$$

and

$$E\left(\chi^2_{n-i}\right)^{-r/2}=2^{-r/2}\frac{\Gamma((n-i-r)/2)}{\Gamma((n-i)/2)}$$

Thus, the rth moment of $\widehat{NMC}_p$ can be written as

$$E(\widehat{NMC}_p)^r=\frac{1}{b_r}(NMC_p)^r$$

where

$$b_r=\left(\frac{2}{n-1}\right)^{rv/2}\prod_{i=1}^{v}\left(\frac{\Gamma((n-i)/2)}{\Gamma((n-i-r)/2)}\right)$$

Based on the definition of $\widehat{NMC}_{pm}$, the rth moment of $\widehat{NMC}_{pm}$ can be written as

$$E(\widehat{NMC}_{pm})^r=|\mathbf{A}^*|^{-r/2}E(|\mathbf{S}^*|^{-r/2})$$

By Theorem 3 of Dahel and Giri[13], one can obtain

$$E(|\mathbf{S}^*|)^{-r/2}=\left(\frac{|\boldsymbol{\Sigma}|}{(n-1)^v}\right)^{-r/2}E(\chi^2_n(\lambda))^{-r/2}\prod_{i=1}^{v-1}E(\chi^2_{n-i})^{-r/2}$$

Moreover, $\chi^2_n(\lambda)$ distribution can be written as a mixture of central χ^2_{n+2j} distributions with Poisson weights

$$\mathrm{e}^{-\lambda/2}\frac{(\lambda/2)^j}{j!}$$

Then,

$$E(\chi_n^2(\lambda))^{-r/2} = e^{-\lambda/2} \sum_{j=0}^{\infty} \frac{(\lambda/2)^j}{j!} E(\chi_{n+2j}^2)^{-r/2}$$

(see Pearn *et al.*[14]) and the rth moment of $\widehat{NMC}_{pm}$ is given by

$$E(\widehat{NMC}_{pm})^r = \frac{1}{b_r^*}(\widehat{NMC}_{pm})^r$$

where

$$b_r^* = \left(\frac{2^{v-1}}{(n-1)^v(1+\lambda/n)}\right)^{r/2} e^{\lambda/2} \left(\sum_{j=0}^{\infty} \frac{(\lambda/2)^j \Gamma((n+2j-r)/2)}{j!\,\Gamma((n+2j)/2)}\right)^{-1} \prod_{i=1}^{v-1} \left(\frac{\Gamma((n-i)/2)}{\Gamma((n-i-r)/2)}\right)$$

Appendix C: Derivation of the $100(1-\alpha)\%$ confidence intervals for NMC_p and NMC_{pm}

According to Maman[12], one can obtain that $|\mathbf{S}|/|\Sigma|$ is the distribution of $W = \prod_{i=1}^{v} \chi_{n-i}^2/(n-1)^v$. Based on the definitions of NMC_p and $\widehat{NMC}_p$, the ratio $NMC_p/\widehat{NMC}_p$ is equal to $(|\mathbf{S}|/|\Sigma|)^{1/2}$. Furthermore, we let w_α be a constant such that $Pr\{W > w_\alpha\} = 1-\alpha$, then the following properties hold:

$$Pr\{w_{\alpha/2} < W < w_{1-\alpha/2}\} = 1-\alpha$$

$$Pr\{w_{\alpha/2} < |\mathbf{S}|/|\Sigma| < w_{1-\alpha/2}\} = 1-\alpha$$

$$Pr\{\sqrt{w_{\alpha/2}} < NMC_p/\widehat{NMC}_p < \sqrt{w_{1-\alpha/2}}\} = 1-\alpha$$

$$Pr\{\widehat{NMC}_p\sqrt{w_{\alpha/2}} < NMC_p < \widehat{NMC}_p\sqrt{w_{1-\alpha/2}}\} = 1-\alpha$$

Hence, a $100(1-\alpha)\%$ confidence interval for the NMC_p index is given by

$$[\widehat{NMC}_p\sqrt{w_{\alpha/2}},\ \widehat{NMC}_p\sqrt{w_{1-\alpha/2}}]$$

By Theorem 3 of Dahel and Giri[13], one can obtain that $|\mathbf{S}^*|/|\Sigma|$ is the distribution of $\chi_n^2(\lambda)\prod_{i=1}^{v-1}\chi_{n-i}^2/(n-1)^v$. Based on the definitions of NMC_{pm} and $\widehat{NMC}_{pm}$, the ratio $NMC_{pm}/\widehat{NMC}_{pm}$ is equal to $(|\mathbf{S}^*|/|\Sigma^*|)^{1/2}$. Furthermore, we let w_α^* be a constant such that $Pr\{W^* > w_\alpha^*\} = 1-\alpha$. Then, the following properties hold:

$$Pr\{w_{\alpha/2}^* < W^* < w_{1-\alpha/2}^*\} = 1-\alpha$$

$$Pr\left\{\frac{w_{\alpha/2}^*}{(1+\lambda/n)} < \frac{|\mathbf{S}^*|}{|\Sigma^*|} < \frac{w_{1-\alpha/2}^*}{(1+\lambda/n)}\right\} = 1-\alpha$$

$$Pr\left\{\sqrt{\frac{w_{\alpha/2}^*}{(1+\lambda/n)}} < \frac{NMC_{pm}}{\widehat{NMC}_{pm}} < \sqrt{\frac{w_{1-\alpha/2}^*}{(1+\lambda/n)}}\right\} = 1-\alpha$$

$$Pr\left\{\widehat{NMC}_{pm}\sqrt{\frac{w_{\alpha/2}^*}{(1+\lambda/n)}} < NMC_{pm} < \widehat{NMC}_{pm}\sqrt{\frac{w_{1-\alpha/2}^*}{(1+\lambda/n)}}\right\} = 1-\alpha$$

Hence, a $100(1-\alpha)\%$ confidence interval for the NMC_{pm} index is given by

$$\left[\widehat{NMC}_{pm}\sqrt{\frac{w_{\alpha/2}^*}{1+\lambda/n}},\ \widehat{NMC}_{pm}\sqrt{\frac{w_{1-\alpha/2}^*}{1+\lambda/n}}\right]$$

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