Non-decreasing Degradation Data Analysis on Gamma Process

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	Research areas	Research work
Research	1. Bayesian inference 2. Structural equation modeling 3. Time series	SEM $\xrightarrow{\text{time series data}}$ Bayesian + SEM SEM $\xrightarrow{\text{longitudinal data}}$ Bayesian + SEM JSPI SIM prior → empirical Bayesian $\xrightarrow{\text{THSV}}$ JSCS prior → mixture prior $\xrightarrow{\text{LMM}}$ JAS
	4. Reliability analysis 5. Reparable system 6. Optimal design 7. Degradation analysis	ALT → CSALT vs SSALT $\xrightarrow{\text{optimal comparison}}$ QREI TRP + degradation $\xrightarrow{\text{optimal}}$ JCSA EOP prediction of battery → JQT ALT + Competing Risk → IEEE

Ongoing research:

Bayesian Model Selection:

mixture prior

Degradation Analysis:

trend gamma process (TGP),

trend inverse Gaussian process (TIGP),

heterogeneity.

non-decreasing degradation analysis.

Optimal Design:

optimal design for TGP, TIGP.

EOP prediction of battery:

NASA battery, voltage decay data.

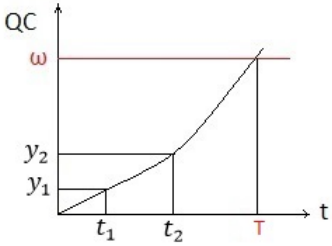
Outline

Introduction and Motivation

- Reliability: reliability is the probability that a product can work.
- Nowadays, for the highly reliable products, it is difficult to obtain sufficient failure information (within a reasonable experiment time period) to predict the reliability information.

- In this situation, **degradation analysis** is an useful tool to predict the product's lifetime precisely, which is a methodology of analyzing the product's quality characteristics (QCs) to make a prediction, while the QC is highly related to the reliability information and degrades over time. (eg. light for LED)

- In degradation analysis, the product's lifetime can be denoted by the first passage time when the degradation path crosses the pre-specific failure threshold.



- The stochastic process is usually used to model the observed QC or degradation data. The widely used stochastic processes are:

  Wiener process - the increments of degradation path are assumed to be independent **normal distribution**. (Doksum and Hoyland, 1992; Whitemore, 1995; Whitemore and Schenkelberg, 1997)

  Gamma process - the increments of degradation path are assumed to be independent **gamma distribution**. (Berman, 1981; Lawless and Crowder, 2004; Park and Padgett, 2005; Wang, 2008; Pan and Balakrishnan, 2011)

- Gamma process (GP) is suitable for modeling the monotonic degradation path. (strictly increasing or decreasing)

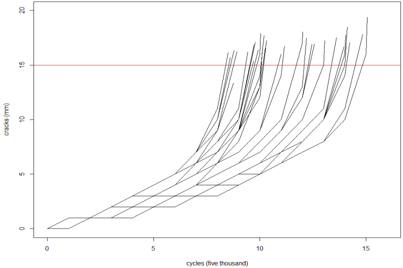
- For the monotonic degradation path, some reasons may cause that the degradation path appears the case of **zero-increment**. For example,

  * when the experimenter want to obtain more data information within a specified experiment period, they may increase the measuring frequency up
  * when the measuring instrument is too old to lack of accuracy, the numbers of decimal digits of observations would be not enough

  Such situations might lead to the occurrence of zero-increment in the experiment. That is, the degradation path has a **non-decreasing property**.

- In this situation, GP model **can not** be applied to the non-decreasing degradation data because of the definition of the GP model whose increments of degradation path following a gamma distribution.

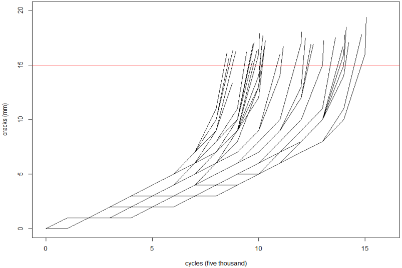
- Fatigue degradation path from Wu and Ni (2003).

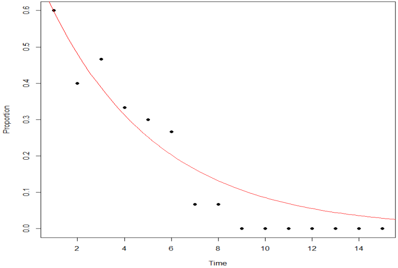


- The goal of this study is construct a degradation model to deal with the non-decreasing degradation data.

- We attempt to extend the GP model incorporated the mixture distribution. Assume the increment of degradation path follows a mixture distribution with a p chance of following a degenerate distribution at zero, and a $(1-p)$ chance of having a gamma distribution, whic is so called the **non-decreasing gamma process (NGP) model**.

- Furthermore, we find that the non-decreasing phenomenon usually occurs in the early period of the experiment and the decay will be violent with the experiment tested time.
- See the fatigue degradation path and proportion of zero-increment from Wu and Ni (2003).





- Motivated by this situation, the probability p in the NGP model should not be a constant. That is, the the probability p is non-homogeneous. Such that, we treat the probability p as a function of time t, namely $p(t)$, to construct the **non-homogeneously non-decreasing gamma process (NNGP) model**.

Non-decreasing Gamma Process (NGP) Model

GP$(\alpha(t), \beta)$

Let $Y(t_j)$ be QC at time t_j, $j = 1, \ldots, m$, where m is the total measurements. Suppose $\{Y(t)\}$ follows the **GP model**, then

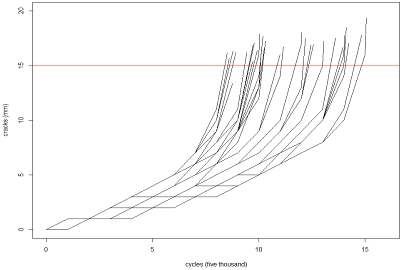
1. $Y(0) = 0$,
2. $\Delta Y(t_j) = Y(t_j) - Y(t_{j-1}) \sim Ga(\alpha(t_j) - \alpha(t_{j-1}), \beta)$,
3. $\{Y(t)\}$ has independent increment.

Let observations $y(t_j) = y_j, \forall j$, then the likelihood function of GP$(\alpha(t), \beta)$ model can be expressed as

$$\prod_{j=1}^{m} \frac{\Delta y_j^{\Delta\alpha_j - 1} e^{-\Delta y_j/\beta}}{\beta^{\Delta\alpha_j}\Gamma(\Delta\alpha_j)},$$

where $\Delta y_j = y_j - y_{j-1}$ and $\Delta\alpha_j = \alpha(t_j) - \alpha(t_{j-1})$.

- Fatigue degradation path from Wu and Ni (2003).



*Measuring frequency is too high (interval arrival time is too short).
*Measuring instrument is not precise.

- The mixture distribution is used to construct the **non-decreasing gamma process (NGP) model**.

NGP$(\alpha(t), \beta, p)$

Suppose the degradation process $Y(t)$ follows the **NGP model**, then

1. $Y(0) = 0$,
2. $Z_j = Y(t_j) - Y(t_{j-1}) \sim f_j{}^N(z)$,
3. $\{Y(t)\}$ has independent increment,

where Z_j follow a mixture density function $f_j{}^N(z)$ with distribution

$$f_j{}^N(z) = \boldsymbol{p} I_{\{z=0\}} + \boldsymbol{(1-p)} \frac{z^{\Delta\alpha_j - 1} e^{-z/\beta}}{\beta^{\Delta\alpha_j}\Gamma(\Delta\alpha_j)} I_{\{z>0\}},$$

and $\Delta\alpha_j = \alpha(t_j) - \alpha(t_{j-1})$.

Suppose the observations $\{y_j\}_{j=1}^m$ follow the NGP$(\alpha(t), \beta, p)$ model with $z_j = y_j - y_{j-1}$. Let δ_j be an indicator function with

$$\delta_j = \begin{cases} 1, & z_j > 0, \\ 0, & z_j = 0. \end{cases}$$

Thus, the likelihood function of NGP$(\alpha(t), \beta, p)$ model can be expressed as

$$\prod_{j=1}^{m} p^{1-\delta_j} \left[(1-p) \, \frac{z_j^{\Delta\alpha_j - 1} e^{-z_j/\beta}}{\beta^{\Delta\alpha_j} \Gamma(\Delta\alpha_j)} \right]^{\delta_j}.$$

Suppose n degradation paths are independent each other. Let $y_{i,j}$ represent the observation at time $t_{i,j}$, for $j = 1, \ldots, m_i$, $i = 1, \ldots, n$ with $z_{i,j} = y_{i,j} - y_{i,j-1}$. Define

$$\delta_{i,j} = \begin{cases} 1, & z_{i,j} > 0, \\ 0, & z_{i,j} = 0. \end{cases}$$

Suppose the observation $\{y_{i,j}, \forall i, j\}$ follow the NGP $(\alpha(t), \beta, p)$, then the total likelihood function can be derived as

$$L(\boldsymbol{\theta}|\boldsymbol{y}) = \prod_{i=1}^{n} \prod_{j=1}^{m_i} p^{1-\delta_{i,j}} \left[(1-p) \frac{z_{i,j}^{\Delta\alpha_{i,j}-1} e^{-z_{i,j}/\beta}}{\beta^{\Delta\alpha_{i,j}} \Gamma(\Delta\alpha_{i,j})} \right]^{\delta_{i,j}},$$

where $\boldsymbol{y} = \{y_{i,j}, i = 1, \ldots, n, j = 1, \ldots, m_i\}$, $\Delta\alpha_{i,j} = \alpha(t_{i,j}) - \alpha(t_{i,j-1})$.

- Consider $\alpha(t) = at^b$. Hence, $\boldsymbol{\theta} = (a, b, \beta, p)$ for the NGP (at^b, β, p) model. The MLEs for $\boldsymbol{\theta}$ can be obtained as follows.

  p: analytical form
  β: analytical form
  a: numerical optimization
  b: numerical optimization

- The analytical forms of $\hat{p}$ and $\hat{\beta}$ can be derived as

$$\hat{p} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m_i} (1 - \delta_{i,j})}{\sum_{i=1}^{n} m_i}$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m_i} \delta_{i,j} z_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{m_i} \delta_{i,j} \Delta\hat{\alpha}_{i,j}}.$$

- The **reliability inference** for the NGP$(\alpha(t), \beta, p)$:

  For the degradation process $\{Y(t)\}$, the product's lifetime T is denoted by the first passage time of crossing the pre-specific failure threshold ω, then

$$T = \text{inf}\{t | Y(t) \geqslant \omega\}$$

- Suppose $\{Y(t)\}$ follows the NGP$(\alpha(t), \beta, p)$, and $Z_j = Y(t_j) - Y(t_{j-1})$, then the corresponding indicator variable δ_j^* is

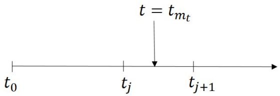
$$\delta_j^* = \begin{cases} 1, & \text{with probability } 1-p, \\ 0, & \text{with probability } p. \end{cases}$$

- The cumulative density function (cdf) of T for the NGP$(\alpha(t),\beta,p)$ is

$$
\begin{aligned}
F_T(t) &= P(T \le t) = P\left(\sum_{j=1}^{m_t} Z_j \ge \omega\right) \\
&= \sum_{k=0}^{m_t}\left\{ P\left(\sum_{j=1}^{m_t} Z_j \ge \omega \,\middle|\, |\boldsymbol{\delta}^*|^2 = k\right) P\left(|\boldsymbol{\delta}^*|^2 = k\right)\right\} \\
&= \sum_{k=0}^{m_t}\left\{ \sum_{\substack{i_1,i_2,\ldots,i_k\in\{1,2,\ldots,m_t\}\\ |\boldsymbol{\delta}^*|^2=k}} \Gamma_I\left(\frac{\omega}{\beta}, \sum_{j=1}^{k}\Delta\alpha(t_{i_j})\right)(1-p)^k p^{m_t-k}\right\}.
\end{aligned}
$$

For given $t>0$, we have $m_t = \sup\{j | t_j < t\} + 1$, $t_{m_t} = t$, $Z_{m_t} = Y(t) - Y(t_{m_t-1})$ and $\delta^*_{m_t} \sim \mathrm{Ber}(1-p)$ on interval $(t_{m_t-1}, t\,]$.

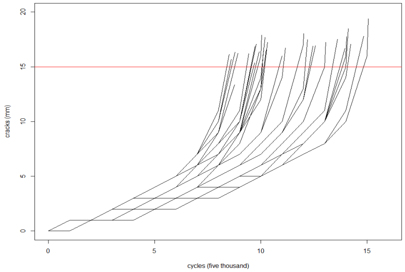
$|\boldsymbol{\delta}^*|^2 = \sum_{j=1}^{m_t} \delta_j^{*2}$, $\Gamma_I(s,t) = \int_s^\infty y^{t-1}e^{-y}dy$, $\Delta\alpha(t_j) = \alpha(t_j) - \alpha(t_{j-1})$.

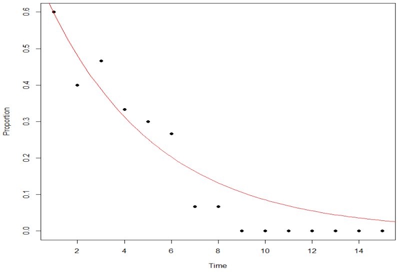


- The q^{th} quantile of T can be numerically obtained via $t_q = F_T^{-1}(q)$.
- The MTTF (mean-time-to-failure) can be obtained by the Monte Carlo method (Meeker and Escobar, 1998).

Non-homogeneously Non-decreasing Gamma Process (NNGP) Model

- Fatigue degradation path and proportion of zero-increment.





  * Measuring frequency is too high.
  * Measuring instrument is not precise.
  * The non-decreasing phenomenon usually occurs in the early period.
  * The decay will be violent with the experiment tested time.

- We take the probability p to be a function of time t and then construct the **non-homogeneously non-decreasing gamma process (NNGP) model**.

NNGP$(\alpha(t), \beta, p(t))$

Suppose the degradation process $Y(t)$ follows the **NNGP model**, then

1. $Y(0) = 0$,
2. $Z_j = Y(t_j) - Y(t_{j-1}) \sim {f_j}^{NN}(z)$,
3. $\{Y(t)\}$ has independent increment,

where Z_j follows a mixture distribution with

$$ {f_j}^{NN}(z) = \boldsymbol{p(t_j)} I_{\{z=0\}} + \boldsymbol{(1 - p(t_j))} \frac{z^{\Delta\alpha_j - 1} e^{-z/\beta}}{\beta^{\Delta\alpha_j} \Gamma(\Delta\alpha_j)} I_{\{z>0\}}, $$

and $\Delta\alpha_j = \alpha(t_j) - \alpha(t_{j-1})$.

Suppose n degradation paths are independent each other. Let $y_{i,j}$ represent the observation at time $t_{i,j}$, for $j = 1, \ldots, m_i$, $i = 1, \ldots, n$ with $z_{i,j} = y_{i,j} - y_{i,j-1}$. Define

$$\delta_{i,j} = \begin{cases} 1, & z_{i,j} > 0, \\ 0, & z_{i,j} = 0. \end{cases}$$

Suppose the observation $\{y_{i,j}, \forall i, j\}$ follow the NNGP $(\alpha(t), \beta, p(t))$, then the total likelihood function can be derived as

$$L(\boldsymbol{\theta}|\boldsymbol{y}) = \prod_{i=1}^{n} \prod_{j=1}^{m_i} (p(t_{i,j}))^{1-\delta_{i,j}} \left[(1 - p(t_{i,j})) \frac{z_{i,j}^{\Delta\alpha_{i,j}-1} e^{-z_{i,j}/\beta}}{\beta^{\Delta\alpha_{i,j}} \Gamma(\Delta\alpha_{i,j})} \right]^{\delta_{i,j}},$$

where $\boldsymbol{y} = \{y_{i,j}, i = 1, \ldots, n, j = 1, \ldots, m_i\}$, $\Delta\alpha_{i,j} = \alpha(t_{i,j}) - \alpha(t_{i,j-1})$.

- Consider $\alpha(t) = at^b$ and $\boldsymbol{p(t) = ch^t}$ (depend on case study), the unknown parameter vector for the NNGP (at^b, β, ch^t) model is $\boldsymbol{\theta} = (a, b, \beta, c, h)$. The MLEs for $\boldsymbol{\theta}$ can be obtained as follows.

  β: analytical form
  c: solving a equation
  $\{h,\ a,\ b\}$: numerical optimization

- $\hat{\beta}$ and $\hat{c}$ can be obtained by

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\delta_{i,j}}{1 - \hat{c}h^{t_{i,j}}},$$

$$\hat{\beta} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m_i} \delta_{i,j} z_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{m_i} \delta_{i,j} \Delta\hat{\alpha}_{i,j}},$$

where $\Delta\hat{\alpha}_{i,j} = \hat{a}(t_{i,j}^{\hat{b}} - t_{i,j-1}^{\hat{b}})$.

- The **reliability inference** for the NNGP $(\alpha(t), \beta, p(t))$:

  The product's lifetime T is denoted by

$$T = \inf\{t | Y(t) \geqslant \omega\}$$

- Suppose $\{Y(t)\}$ follows the NNGP$(\alpha(t), \beta, p(t))$, and $Z_j = Y(t_j) - Y(t_{j-1})$, for $j = 1, \ldots, m$, then the corresponding indicator function $\delta_j^{'}$ is

$$\delta_j' = \begin{cases} 1, & \text{with probability } 1 - p(t_j), \\ 0, & \text{with probability } p(t_j). \end{cases}$$

- The cdf of T for the NNGP$(\alpha(t), \beta, p(t))$ is

$$\begin{aligned} F_T(t) &= P(T \le t) = P\left(\sum_{j=1}^{m_t} Z_j \ge \omega\right) \\ &= \sum_{k=0}^{m_t}\left\{P\left(\sum_{j=1}^{m_t} Z_j \ge \omega \middle| \, |\boldsymbol{\delta'}|^2 = k\right) P\left(|\boldsymbol{\delta'}|^2 = k\right)\right\} \\ &= \sum_{k=0}^{m_t}\left\{\sum_{\substack{i_1, i_2, \dots, i_k \in \{1,2,\dots,m_t\} \\ |\boldsymbol{\delta'}|^2 = k}} \Gamma_I\left(\frac{\omega}{\beta}, \sum_{j=1}^{k} \Delta\alpha(t_{i_j})\right) \prod_{j=1}^{k}(1 - p(t_{i_j})) \prod_{\substack{s=1 \\ s \ne i_1, i_2, \dots i_k}}^{m_t} p(t_s)\right\}. \end{aligned}$$

where $\delta'_{m_t} \sim \mathrm{Ber}(1 - p(t))$ on interval $(t_{m_t - 1}, t\ ]$ and $|\boldsymbol{\delta'}|^2 = \sum_{j=1}^{m_t} \delta'^2_j$.

- The q^{th} quantile of T can be numerically obtained via $t_q = F_T^{-1}(q)$.

- The MTTF (mean-time-to-failure) can be obtained by the Monte Carlo method (Meeker and Escobar, 1998).

Simulation Study

NGP Simulation:

- For the NGP (at^b, β, p),

1. $(a, b, \beta, p) = (2,\ 1.28, 0.36, 0.05)$
2. sample size $n = 30$
3. measured frequency $m = 15$
4. pre-specified threshold $\omega = 15$
5. quantile t_q under q=0.01, 0.05, 0.1, 0.5
6. bootstrap resampling number $B = 2000$
7. 100 simulation runs

Table 1: The results of 100 simulation runs for data from the NGP model.

Model	True value	Estimate	SD	95% C.I.
NGP	$a = 2$	2.0370	0.2141	$[1.5500, 2.7374]$
	$b = 1.28$	1.2807	0.0316	$[1.1941, 1.3701]$
	$\beta = 0.36$	0.3576	0.0301	$[0.2793, 0.4410]$
	$p = 0.05$	0.0495	0.0117	$[0.0209, 0.0849]$
	$t_{0.01} = 8.0039$	7.9913	0.2771	$[7.2748, 8.7091]$
	$t_{0.05} = 8.9060$	8.8843	0.2667	$[8.1834, 9.5810]$
	$t_{0.1} = 9.4010$	9.3727	0.2647	$[8.6737, 10.0581]$
	$t_{0.5} = 11.2135$	11.1723	0.2723	$[10.4316, 11.8992]$
	MTTF$= 11.2639$	11.1901	0.4662	$[10.3179, 12.1343]$

NNGP Simulation:

- For the NNGP (at^b, β, ch^t),

1. $(a, b, \beta, c, h) = (2, 1.28, 0.36, 0.77, 0.78)$
2. sample size $n = 30$
3. measured frequency $m = 15$
4. pre-specified threshold $\omega = 15$
5. quantile t_q under q=0.01, 0.05, 0.1, 0.5
6. bootstrap resampling number $B = 2000$
7. 100 simulation runs

Table 2: The results of 100 simulation runs for data from the NNGP model.

Model	True value	Estimate	SD	95% C.I.
NNGP	$a = 2$	2.0828	0.2561	$[1.4995, 2.9520]$
	$b = 1.28$	1.2712	0.0331	$[1.1744, 1.3730]$
	$\beta = 0.36$	0.3574	0.0314	$[0.2792, 0.4460]$
	$c = 0.77$	0.7702	0.0992	$[0.5205, 1.0650]$
	$h = 0.78$	0.7785	0.0249	$[0.7025, 0.8442]$
	$t_{0.01} = 8.9144$	8.9037	0.2469	$[8.1489, 9.6258]$
	$t_{0.05} = 9.9235$	9.9136	0.2356	$[9.1796, 10.6085]$
	$t_{0.1} = 10.4771$	10.4657	0.2337	$[9.7351, 11.1475]$
	$t_{0.5} = 12.4717$	12.4685	0.2443	$[11.7000, 13.1828]$
	MTTF$= 12.4980$	12.4825	0.4908	$[11.5486, 13.4182]$

Comparison Analysis:

- Generate GP, NGP and NNGP models simulation data to apply to NGP and NNGP models.

1. parameter settings are given in the following tables
2. sample size $n = 30$
3. measured frequency $m = 15$
4. pre-specified threshold $\omega = 15$
5. quantile t_q under q=0.01, 0.05, 0.1, 0.5
6. bootstrap resampling number $B = 2000$
7. 100 simulation runs

Table 3: The results of 100 simulation runs for data from the GP model.

NGP model				NNGP model			
True value	Estimate	SD	95% CI	True value	Estimate	SD	95% CI
$a = 2$	2.0539	0.1868	$[1.5698, 2.6825]$	$a = 2$	2.0522	0.1870	$[1.5675, 2.6820]$
$b = 1.28$	1.2750	0.0321	$[1.1887, 1.3656]$	$b = 1.28$	1.2762	0.0321	$[1.1901, 1.3673]$
$\beta = 0.36$	0.3550	0.0218	$[0.2893, 0.4281]$	$\beta = 0.36$	0.3542	0.0216	$[0.2882, 0.4259]$
$p = 0$	0	0	$[0, 0]$	$c = 0$	0	0	$[0, 0]$
				h	0.8699	0.3380	$[0.0000, 0.9999]$
$t_{0.01} = 7.8436$	7.8873	0.2045	$[7.2739, 8.5167]$	$t_{0.01} = 7.8436$	7.8929	0.2038	$[7.2855, 8.5288]$
$t_{0.05} = 8.6923$	8.7360	0.1982	$[8.1451, 9.3394]$	$t_{0.05} = 8.6923$	8.7403	0.1977	$[8.1528, 9.3502]$
$t_{0.1} = 9.1496$	9.1935	0.1964	$[8.6109, 9.7891]$	$t_{0.1} = 9.1496$	9.1970	0.1960	$[8.6164, 9.7978]$
$t_{0.5} = 10.7890$	10.8343	0.2013	$[10.2443, 11.4316]$	$t_{0.5} = 10.7890$	10.8348	0.2013	$[10.2429, 11.4368]$
MTTF$= 10.7974$	10.8779	0.3844	$[10.0980, 11.6050]$	MTTF$= 10.7974$	10.8690	0.3470	$[10.0989, 11.6050]$

Table 4: The results of 100 simulation runs for data from the NGP model with $p = 0.05$.

NGP model				NNGP model			
True value	Estimate	SD	95% CI	True value	Estimate	SD	95% CI
$a = 2$	2.0370	0.2141	$[1.5500, 2.7374]$	$a = 2$	2.0357	0.2147	$[1.5453, 2.7400]$
$b = 1.28$	1.2807	0.0316	$[1.1941, 1.3701]$	$b = 1.28$	1.2818	0.0317	$[1.1957, 1.3727]$
$\beta = 0.36$	0.3576	0.0301	$[0.2793, 0.4410]$	$\beta = 0.36$	0.3569	0.0299	$[0.2780, 0.4384]$
$p = 0.05$	0.0495	0.0117	$[0.0209, 0.0849]$	$c = 0.05$	0.0611	0.0212	$[0.0256, 0.1610]$
				$h = 1$	0.9697	0.0420	$[0.7907, 0.9999]$
$t_{0.01} = 8.0039$	7.9913	0.2771	$[7.2748, 8.7091]$	$t_{0.01} = 8.0039$	8.0041	0.2750	$[7.3007, 8.7240]$
$t_{0.05} = 8.9060$	8.8843	0.2667	$[8.1834, 9.5810]$	$t_{0.05} = 8.9060$	8.8946	0.2649	$[8.2048, 9.5918]$
$t_{0.1} = 9.4010$	9.3727	0.2647	$[8.6737, 10.0581]$	$t_{0.1} = 9.4010$	9.3810	0.2630	$[8.6912, 10.0675]$
$t_{0.5} = 11.2135$	11.1723	0.2723	$[10.4316, 11.8992]$	$t_{0.5} = 11.2135$	11.1676	0.2731	$[10.4231, 11.8828]$
MTTF$= 11.2639$	11.1901	0.4662	$[10.3179, 12.1343]$	MTTF$= 11.2639$	11.2284	0.4659	$[10.3090, 12.1093]$

Table 5: The results of 100 simulation runs for data from the NGP model with $p = 0.1$.

NGP model				NNGP model			
True value	Estimate	SD	95% CI	True value	Estimate	SD	95% CI
$a = 2$	1.9859	0.1895	$[1.5277, 2.6483]$	$a = 2$	1.9844	0.1898	$[1.5211, 2.6473]$
$b = 1.28$	1.2817	0.0267	$[1.1998, 1.3650]$	$b = 1.28$	1.2828	0.0268	$[1.2009, 1.3672]$
$\beta = 0.36$	0.3647	0.0287	$[0.2888, 0.4491]$	$\beta = 0.36$	0.3640	0.0286	$[0.2876, 0.4473]$
$p = 0.1$	0.0957	0.0152	$[0.0565, 0.1387]$	$c = 0.1$	0.1278	0.0356	$[0.0658, 0.2582]$
				$h = 1$	0.9670	0.0310	$[0.8512, 0.9999]$
$t_{0.01} = 8.1899$	8.1453	0.2599	$[7.4444, 8.8891]$	$t_{0.01} = 8.1899$	8.1935	0.2693	$[7.4918, 8.9652]$
$t_{0.05} = 9.1478$	9.1012	0.2585	$[8.4146, 9.8422]$	$t_{0.05} = 9.1478$	9.1519	0.2702	$[8.4577, 9.9265]$
$t_{0.1} = 9.6812$	9.6313	0.2586	$[8.9365, 10.3823]$	$t_{0.1} = 9.6812$	9.6810	0.2715	$[8.9750, 10.4720]$
$t_{0.5} = 11.6890$	11.6260	0.2957	$[10.8596, 12.4735]$	$t_{0.5} = 11.6890$	11.6650	0.3133	$[10.8784, 12.5540]$
MTTF$= 11.7706$	11.7145	0.5521	$[10.7360, 12.7717]$	MTTF$= 11.7706$	11.7661	0.5055	$[10.7527, 12.8192]$

Table 6: The results of 100 simulation runs for data from the NGP model with $p = 0.3$.

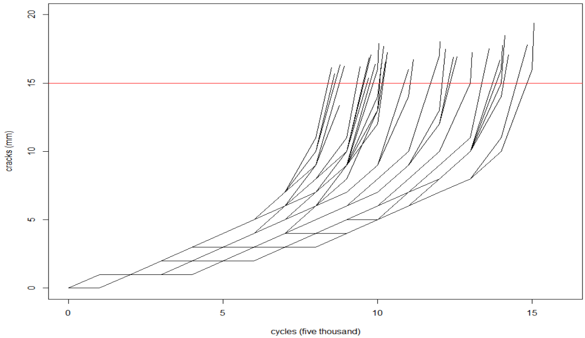
NGP model				NNGP model			
True value	Estimate	SD	95% CI	True value	Estimate	SD	95% CI
$a = 2$	2.0376	0.2384	$[1.4675, 2.7912]$	$a = 2$	2.0375	0.2384	$[1.4683, 2.7914]$
$b = 1.28$	1.2801	0.0330	$[1.1901, 1.3749]$	$b = 1.28$	1.2805	0.0330	$[1.1904, 1.3756]$
$\beta = 0.36$	0.3558	0.0310	$[0.2748, 0.4441]$	$\beta = 0.36$	0.3556	0.0310	$[0.2741, 0.4428]$
$p = 0.3$	0.2981	0.0227	$[0.2353, 0.3594]$	c=0.3	0.3200	0.0418	$[0.2318, 0.4714]$
				$h = 1$	0.9909	0.0113	$[0.9459, 0.9999]$
$t_{0.01} = 9.1740$	9.2103	0.3615	$[8.3021, 10.2067]$	$t_{0.01} = 9.1740$	9.2480	0.3674	$[8.3479, 10.2895]$
$t_{0.05} = 10.4780$	10.4992	0.3780	$[9.5339, 11.5559]$	$t_{0.05} = 10.4780$	10.5285	0.3936	$[9.5529, 11.6390]$
$t_{0.1} = 11.2162$	11.2437	0.3982	$[10.2246, 12.3500]$	$t_{0.1} = 11.2162$	11.2662	0.4177	$[10.2231, 12.4334]$
$t_{0.5} = 14.2348$	14.2534	0.5053	$[12.9412, 15.6857]$	$t_{0.5} = 14.2348$	14.2117	0.5457	$[12.8261, 15.7081]$
MTTF$= 14.4487$	14.3647	0.8947	$[12.8565, 16.2472]$	MTTF$= 14.4487$	14.4421	0.8127	$[12.7443, 16.2090]$

Table 7: The results of 100 simulation runs for data from the NNGP model.

NGP model				NNGP model			
True value	Estimate	SD	95% CI	True value	Estimate	SD	95% CI
$a = 2$	2.0852	0.2559	$[1.5451, 2.8938]$	$a = 2$	2.0828	0.2561	$[1.4995, 2.9520]$
$b = 1.28$	1.2701	0.0331	$[1.1819, 1.3614]$	$b = 1.28$	1.2712	0.0331	$[1.1744, 1.3730]$
$\beta = 0.36$	0.3581	0.0315	$[0.2802, 0.4475]$	$\beta = 0.36$	0.3574	0.0314	$[0.2792, 0.4460]$
p	0.2017	0.0169	$[0.1504, 0.2536]$	$c = 0.77$	0.7702	0.0992	$[0.5205, 1.0650]$
				$h = 0.78$	0.7785	0.0249	$[0.7025, 0.8442]$
$t_{0.01} = 8.9144$	8.6143	0.2556	$[7.8376, 9.3793]$	$t_{0.01} = 8.9144$	8.9037	0.2469	$[8.1489, 9.6258]$
$t_{0.05} = 9.9235$	9.7236	0.2424	$[8.9503, 10.4891]$	$t_{0.05} = 9.9235$	9.9136	0.2356	$[9.1796, 10.6085]$
$t_{0.1} = 10.4771$	10.3561	0.2463	$[9.5827, 11.1213]$	$t_{0.1} = 10.4771$	10.4657	0.2337	$[9.7351, 11.1475]$
$t_{0.5} = 12.4717$	12.8388	0.2701	$[11.9204, 13.7657]$	$t_{0.5} = 12.4717$	12.4685	0.2443	$[11.7000, 13.1828]$
MTTF$= 12.4980$	13.0966	0.7164	$[11.8063, 14.2397]$	MTTF$= 12.4980$	12.4825	0.4908	$[11.5486, 13.4182]$

Case Study

- The real data is fatigue crack growth data taken from Wu and Ni (2003). The observed QC was the crack size (mm). There were 30 tested samples for fatigue crack development, and each measurement was recorded every 5,000 cycles. The initial crack size was 18 mm and the failure threshold was determined to be 33 mm. By taking $y^*_{i,j} = y_{i,j} - 18$, then the new threshold is $\omega = 15$ mm.



- For the function $p(t)$, we attempted to fit the proportions via the **power law function** $p(t) = ct^h$ or **log-linear function** $p(t) = ch^t$. Based on the R^2 $(84.84\% > 72.34\%)$, we adopted the log-linear function $p(t) = ch^t$ to the NNGP (at^b, β, ch^t) model.

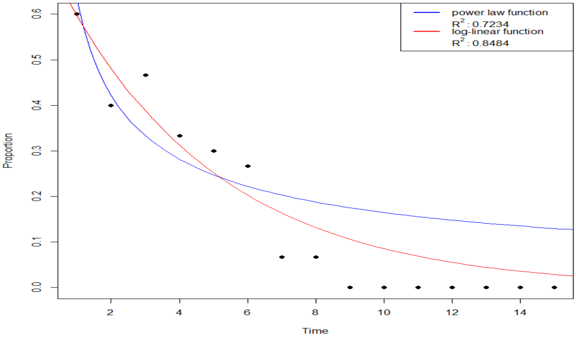


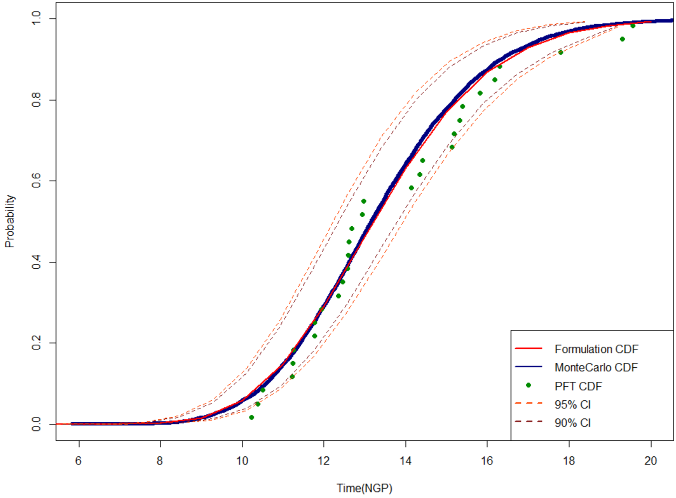
Table 8: The results of NGP and NNGP models for Wu and Ni (2003) fatigue data.

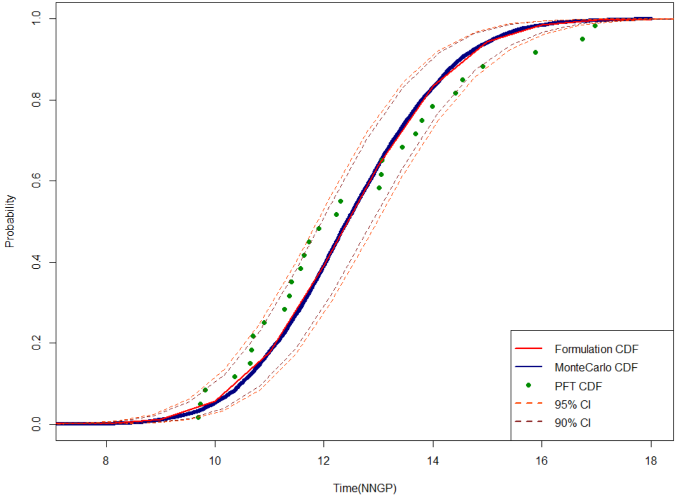
NGP model				NNGP model			
$\boldsymbol{\theta}$	MLE	95% CI	90% CI	$\boldsymbol{\theta}$	MLE	95% CI	90% CI
a	2.0095	$[1.6185, 2.5015]$	$[1.6750, 2.4310]$	a	2.0000	$[1.5613, 2.5844]$	$[1.6363, 2.4943]$
b	1.2858	$[1.2248, 1.3516]$	$[1.2317, 1.3402]$	b	1.2870	$[1.2107, 1.3635]$	$[1.2241, 1.3527]$
β	0.3593	$[0.3030, 0.4250]$	$[0.3116, 0.4165]$	β	0.3560	$[0.2988, 0.4119]$	$[0.3067, 0.4026]$
p	0.2438	$[0.1995, 0.2830]$	$[0.2074, 0.2770]$	c	0.7759	$[0.5663, 0.9762]$	$[0.5893, 0.9502]$
				h	0.7830	$[0.7284, 0.8322]$	$[0.7386, 0.8239]$
$t_{0.01}$	8.7380	$[7.9966, 9.2267]$	$[8.1052, 9.1267]$	$t_{0.01}$	8.9108	$[8.3836, 9.4519]$	$[8.4592, 9.3618]$
$t_{0.05}$	9.8894	$[9.1428, 10.4267]$	$[9.2412, 10.2976]$	$t_{0.05}$	9.9104	$[9.3655, 10.4422]$	$[9.4635, 10.3447]$
$t_{0.1}$	10.5666	$[9.7839, 11.1011]$	$[9.8818, 10.9628]$	$t_{0.1}$	10.4587	$[9.9048, 10.9687]$	$[10.0050, 10.8782]$
$t_{0.5}$	13.2085	$[12.2731, 13.9080]$	$[12.3704, 13.7517]$	$t_{0.5}$	12.4359	$[11.8758, 12.9710]$	$[11.9581, 12.8824]$
MTTF	13.1462	$[12.5572, 14.2150]$	$[12.7234, 14.0772]$	MTTF	12.2082	$[11.6654, 13.2492]$	$[11.7792, 13.1318]$
max- $\ln L$		396.8275		max- $\ln L$		427.9180	
AD p-value		0.9600		AD p-value		0.8264	
KS p-value		0.9988		KS p-value		0.9578	
AIC		863.836		AIC		803.655	

- For model checking, we used the pseudo failure time (PFT, Meeker and Escobar (1998)) as the baseline to diagnose the goodness of fit.

Algorithm 7 PFTs for the NGP (NNGP) model

1. Set $i = 0$.
2. Set $i = i+1$. Compute MLE $\hat{\boldsymbol{\theta}}_i$ for the NGP (NNGP) model only based on i^{th} degradation path.
3. Substitute $\hat{\boldsymbol{\theta}}_i$ into Algorithm 3 to obtain PFT t_i^F.
4. Repeat steps 2-3 for n times to obtain the PFTs to be $\{t_1^F, \ldots, t_n^F\}$.


1.0
0.8
0.6
0.4
0.2
0.0
Probability
6
8
10
12
14
16
18
20
Time(NGP)
Formulation CDF
MonteCarlo CDF
PFT CDF
95% CI
90% CI


Probability
0.0
0.2
0.4
0.6
0.8
1.0
8
10
12
14
16
18
Time(NNGP)
Formulation CDF
MonteCarlo CDF
PFT CDF
95% CI
90% CI

- In addition, the likelihood-ratio test (LR-test) is also used to make a further comparison between the NGP and NNGP models. Based on $p(t) = ch^t$, then the hypothesis testing is given by

$$H_0 : h = 1 \text{ (NGP model)} \quad \text{v.s.} \quad H_1 : h \neq 1 \text{ (NNGP model)}.$$

We have

$$\begin{aligned} \chi^2 &= -2\left(\ln \hat{L}_{\mathrm{NGP}} - \ln \hat{L}_{\mathrm{NNGP}}\right) \\ &= -2(396.8275 - 427.9180) = 62.181 > 3.8415 = \chi^2_{0.05}(1) \end{aligned}$$

to reject H_0 (NGP model). We can conclude that the NNGP model is more suitable model to analyze this non-decreasing degradation data.

Conclusion

- For the high-reliability product, degradation analysis is a commonly used tool to predict product's lifetime.
- For the zero-increment degradation path, we propose the non-decreasing degradation model (NGP and NNGP models) to make the reliability inference.
- Simulation study and case study reveal that our proposed models can provide a well the goodness of fit and reliability prediction. The NNGP model can provide a better fit for this motivated non-decreasing degradation data.

- For the future work,

  * unit-to-unit variation (heterogeneity)

  * non-decreasing inverse Gaussian process model

  * non-homogeneously non-decreasing inverse Gaussian process model

  * nonparametric approach for for $p(t)$:
    $(p_1, \ldots, p_m)$ are set to be the unknown parameters

  * measuring threshold for increment:
    set $p_j = P(Y(t_j) - Y(t_{j-1}) \leq \omega_0)$

Thank you!